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Analysis of the numerics of physics–dynamics coupling

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SUMMARY

A methodology for analysing the numerical properties of schemes for coupling physics parametrizations to a dynamical core is presented. As an example of its application, the methodology is used to study four coupling schemes ('explicit', 'implicit', 'split-implicit' and 'symmetrized split-implicit') in the context of a semi-implicit semi-Lagrangian dynamical core. Each coupling scheme is assessed in terms of its numerical stability and of the accuracy of both its transient and steady-state responses. Additionally, the occurrence of spurious, computational resonance is analysed and discussed. It is found that in this respect all four schemes behave similarly. In particular, in the absence of any damping mechanism to control resonance, the time-step restriction needed to avoid spurious resonance is twice as restrictive for time-dependent forcing as for stationary forcing.

KEYWORDS: Computational resonance Forced response Semi-implicit Semi-Lagrangian

1. INTRODUCTION

Physical parametrization packages form an increasingly large and important component of a modern numerical weather and climate prediction model. However, relatively little attention has been given to the details and implications of how such schemes are coupled to the underlying dynamical core of such models. To provide useful insight into this physics–dynamics coupling issue, Caya *et al.* (1998), hereinafter referred to as CLZ98, examined steady-state solutions of numerical discretizations of

$$\frac{\mathrm{d}F(t)}{\mathrm{d}t} + \sigma F(t) = G, \tag{1.1}$$

where σ (β in CLZ98's notation) and G are constants. The variable $F(t)$ is the dependent variable and, in order to simplify the problem, was chosen to be a function of time only. It is a scalar. The problem is intended to give a framework for the analysis of NWP or climate models which have highly coupled equation sets. In this context, F represents a linear normal mode of those equations, and (1.1), as well as its extensions presented later, represents the evolution equations of such forced normal modes (e.g. Daley (1991)). The term G is a constant forcing, chosen to model a constant diabatic heating in a full model. The σF term represents either a damping mechanism (e.g. boundary-layer diffusion) if $\sigma(>0)$ is purely real, or an oscillatory one if σ is purely imaginary, and attention was focused on the latter to represent a semi-implicit discretization of the terms responsible for gravitational oscillations.

CLZ98's canonical problem has the virtue of providing a very useful insight into the physics–dynamics coupling of a model by reducing a particular computational problem to its essence. Obtaining further insight into physics–dynamics coupling issues is however hindered by the very simplicity of the CLZ98 problem. Therefore, here an enhanced canonical problem that subsumes the CLZ98 one is defined and its exact solution given. It includes: (a) spatio-temporal, instead of constant, forcing; (b) horizontal advection (allowing the examination of real and spurious resonance); and (c) the possibility of simultaneously including several parametrized terms, each of which

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may have its own temporal discretization. These generalizations permit the examination of a broader set of numerical physics–dynamics coupling issues while still keeping the analysis tractable.

The use and benefits of the enhanced canonical problem are demonstrated by applying it to the case of a two-time-level semi-implicit semi-Lagrangian discretization of the underlying dynamics. Four strategies for coupling this dynamical model to the physics forcing are considered. These schemes were introduced and analysed in the context of the problem of CLZ98 (modified to allow different discretizations of both oscillatory and damping processes occurring simultaneously) by Staniforth *et al.* (2002). Two of the schemes, referred to here as the 'explicit' and 'split-implicit' schemes, are essentially equivalent to two of the schemes studied by CLZ98 in their simpler problem. Here, an 'implicit' coupling is also examined since, for the canonical problem proposed herein, this coupling is arguably ideal from the perspectives of stability and accuracy, although it is sub-optimal in terms of efficiency. The remaining one, the 'symmetrized split-implicit' coupling, aims to combine the stability and accuracy properties of the implicit coupling with the efficiency of the less-accurate split-implicit one.

A specific issue that is addressed by these applications of the enhanced canonical problem is that of spurious computational resonance. Assuming that the advection term is treated in a semi-Lagrangian manner, previous analyses (e.g., Rivest *et al.* (1994), Côté *et al.* (1995), Héreil and Laprise (1996)) suggest that spurious semi-Lagrangian resonance is likely to occur for stationary, spatially dependent forcing (such as that caused by orography) when the Courant number approximately equals or exceeds unity. Questions which may be asked and which are addressed by the analysis are: 'How serious a problem is spurious resonance due to stationary forcing for the four coupling schemes considered herein?', 'What is the impact of allowing the forcing to vary not only in space but also in time?' and 'How well do the considered schemes handle forcing in general, both constant and spatio-temporally varying forcing?'.

The purpose of the paper is, firstly, to provide a fairly general framework for analysing, tractably, the numerics of physics–dynamics coupling, and, secondly, to illustrate how to exploit this framework by comparing some of the advantages and disadvantages of four possible physics–dynamics couplings. The enhanced canonical problem is defined and discussed in section 2. In section 3, the four physics–dynamics coupling strategies considered herein are defined and applied to the canonical problem. The stability and accuracy of the free, or unforced, solutions for each coupling strategy are analysed in section 4, whilst the non-resonant and resonant forced solutions are discussed in sections 5 and 6 respectively. The effects of semi-Lagrangian interpolation on the analysis are briefly discussed in section 7 and, finally, in section 8 a summary of the results is presented.

2. AN ENHANCED CANONICAL PROBLEM

(*a*) *Definition and discussion*

The canonical problem of CLZ98 is extended here to allow variations in one horizontal spatial dimension, denoted by x, and also to allow the representation of any number (indicated by the summations in (2.1)) of dynamics and physics processes. Therefore, consider the problem

$$\frac{\mathrm{D}\mathcal{F}}{\mathrm{D}t} + i\Big(\sum_j \alpha_j\Big)\mathcal{F} = -\Big(\sum_j \beta_j\Big)\mathcal{F} - i\Big(\sum_j \gamma_j\Big)\mathcal{F} + R(x, t), \qquad (2.1)$$

to be solved subject to an appropriate initial condition, where

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + U\frac{\partial}{\partial x}, \tag{2.2}$$

and α_j, $\beta_j \geqslant 0$, γ_j and $U \geqslant 0$ are all real constants. $R(x, t)$ is a forcing term which is independent of $\mathcal{F}$. The equation is therefore linear in $\mathcal{F}$ and, by the principle of linear superposition, the solution associated with each Fourier mode of R can be considered separately. If, further, the forcing associated with the kth mode oscillates in time with a single frequency Ω_k (multiple frequencies are discussed later), then (2.1) can be simplified to the corresponding problem

$$\frac{\mathrm{D}F}{\mathrm{D}t} + i\Big(\sum_j \alpha_j\Big)F = -\Big(\sum_j \beta_j\Big)F - i\Big(\sum_j \gamma_j\Big)F + R_k\,\mathrm{e}^{i(kx+\Omega_k t)}, \tag{2.3}$$

where k and Ω_k are real constants and R_k is a complex constant. Here k is the horizontal wave-number in the x direction, and R_k and Ω_k are respectively the amplitude and frequency at wave number k of the spatio-temporal forcing. F is that component of $\mathcal{F}$ having wave number k and its x dependency is therefore proportional to $\exp(ikx)$.

Equation (2.3), together with (2.2), is the definition of the enhanced canonical problem proposed here. If $i\left\{\sum_j(\alpha_j + \gamma_j)\right\} + \left(\sum_j \beta_j\right) = \sigma$ and $k = U = \Omega_k = 0$, then this problem reduces to CLZ98's canonical problem (1.1).

The left-hand side of (2.3) models the inviscid dynamics. The $i\alpha_j F$ oscillatory terms represent both the fast dynamical terms that govern the propagation of gravitational and acoustic oscillations and also the Coriolis terms. For numerical stability reasons, these terms are often discretized in a semi-implicit manner, as examined in the CLZ98 analysis. Another, though relatively slow, dynamical time-scale, associated with local advection, is already included via the definition (2.2).

The right-hand side of (2.3) represents parametrized physical forcings and allows the simultaneous examination of various forcings having different time-scales and different temporal discretizations. The $\beta_j F$ and $i\gamma_j F$ terms model parametrizations having, respectively, damping and oscillatory characteristics. For example, setting one β_j equal to $-l^2\nu$, where ν is a constant vertical (horizontal) diffusivity and l is a vertical (horizontal) wave-number, $\beta_j F$ can represent parametrized vertical (horizontal) diffusion effects. This can be a fast time-scale process. Similarly, one of the $i\gamma_j F$ terms can represent fast vertical heat-transport in parametrizations of deep convection, whereas other $\beta_j F$ and $i\gamma_j F$ terms might model respectively further damping and oscillatory parametrizations but, for example, ones that have slow time-scales. The introduction of the time-varying $R_k \exp\{i(kx + \Omega_k t)\}$ term allows the representation of relatively slow, time-dependent, external forcings such as the diurnal variation of radiation, or of very rapid external forcings such as variations in radiation due to rapidly-changing cloud cover and/or cloud properties.

A distinction was made above between fast and slow time-scales because of differing stability considerations (see, e.g., Grabowski and Smolarkiewicz (1996), Wedi (1998), Williamson (1999), Teixeira (2000)). An explicit time-discretization of a slow time-scale process generally has the virtue of simplicity. Also, because the process is slow, an associated stability-limited time-step does not usually hinder computational efficiency. An $O(\Delta t)$ accurate discretization therefore arguably suffices. For a fast time-scale process, an explicit time-discretization of it generally unduly limits the time step because of an over-restrictive stability condition; a more costly time-implicit discretization is therefore usually adopted. However, as illustrated by CLZ98, even though this can address the stability issue, if the resulting discretization is only $O(\Delta t)$ accurate, then the time step may still be unduly limited as a result of time truncation error. This motivates the use of an $O(\Delta t)^2$ accurate time-implicit discretization of such fast processes. In practice, this is difficult to accomplish because the additional complexity associated with the introduction of such parametrized processes gives rise to problems in efficiently coupling them with the time-discretized dynamics. This issue is further discussed in the following sections, and an $O(\Delta t)^2$ accurate time-implicit discretization is described and analysed for the incorporation of, e.g., vertical diffusion.

The enhanced problem proposed here does not depend on time alone but also on space, because of the replacement of $\mathrm{d}F/\mathrm{d}t$ in (1.1) by $\mathrm{D}F/\mathrm{D}t$, defined by (2.2) and the introduction of a spatio-temporal forcing term, $R_k \exp\{i(kx + \Omega_k t)\}$. As a result, a distinct advantage of this canonical problem is that once the analysis is completed, the effects of different forms of external forcing can be trivially examined without the need for reanalysis. In its most general form, the analysis gives the response to, for example, a time- and space-dependent diabatic-heating term. On the other hand, setting both Ω_k and k to zero gives the response to a constant forcing as considered by CLZ98. Setting Ω_k alone to zero, keeping k non-zero, gives the response to a stationary forcing, such as that caused by orography, and this, together with a semi-Lagrangian discretization of $\mathrm{D}F/\mathrm{D}t$, permits the examination of spurious semi-Lagrangian resonance issues, something excluded by the CLZ98 canonical problem.

(*b*) *Exact solution*

In order to keep the analysis relatively compact, in what follows only one dynamical oscillatory process and one physical damping process are considered simultaneously. These are represented respectively by the coefficients $i\alpha$ and β. No oscillatory physical process is considered. Then (2.3) reduces to

$$\frac{\mathrm{D}F}{\mathrm{D}t} + i\alpha F = -\beta F + R_k\, \mathrm{e}^{i(kx+\Omega_k t)}. \tag{2.4}$$

In the analytical solutions which follow, the solution to the richer problem of (2.3) can be obtained by substituting the transformations

$$\alpha = \sum_j (\alpha_j + \gamma_j), \tag{2.5}$$

and

$$\beta = \sum_j \beta_j. \tag{2.6}$$

In the discrete solutions discussed later this is still true, provided each process represented by the transformation is discretized in the same manner.

The exact solution to (2.4) is given by the sum of a free and a forced solution:

$$F_{\text{exact}}(x, t) = F_{\text{exact}}^{\text{free}}(x, t) + F_{\text{exact}}^{\text{forced}}(x, t), \tag{2.7}$$

and these solutions are given below.

(i) *Free solution.* The free solution is that obtained in the absence of any forcing ($R_k \equiv 0$). Seeking a solution of the form $F_{\text{exact}}^{\text{free}}(x, t) = f(t)\, \mathrm{e}^{ikx}$ and solving the resulting first order ODE for $f(t)$, the free solution to (2.4) is given by

$$F_{\text{exact}}^{\text{free}}(x, t) = F_k^{\text{free}}\, \mathrm{e}^{ikx}\, \mathrm{e}^{-(i\alpha+\beta+ikU)t} = F_k^{\text{free}}\, \mathrm{e}^{i\{kx-(\alpha+kU-i\beta)t\}}, \tag{2.8}$$

where F_k^{free} is the initial amplitude of the kth Fourier component of the free solution, given by the initial condition

$$F_{\text{exact}}^{\text{free}}(x, 0) = F_k^{\text{free}} \, \mathrm{e}^{ikx}. \tag{2.9}$$

(ii) *Forced regular solution for* $\beta + i(\alpha + kU + \Omega_k) \neq 0$. When $\beta + i(\alpha + kU + \Omega_k) \neq 0$ the forced component of the solution is given by

$$F_{\text{exact}}^{\text{forced}}(x, t) = F_{\text{exact}}^{\text{regular}}(x, t) \equiv F_k^{\text{regular}} \, \mathrm{e}^{i(kx+\Omega_k t)}, \tag{2.10}$$

where

$$F_k^{\text{regular}} \equiv \frac{R_k}{\beta + i(\alpha + kU + \Omega_k)}, \quad \beta + i(\alpha + kU + \Omega_k) \neq 0, \tag{2.11}$$

is the initial amplitude of the kth Fourier component of the forced regular solution. This corresponds to an initial condition

$$F_{\text{exact}}^{\text{forced}}(x, 0) = F_k^{\text{regular}} \, \mathrm{e}^{ikx}. \tag{2.12}$$

(iii) *Forced resonant solution for* $\beta + i(\alpha + kU + \Omega_k) = 0$. Resonance occurs for the singular case $\beta + i(\alpha + kU + \Omega_k) = 0$ and, noting that β and $(\alpha + kU + \Omega_k)$ are thus both identically zero, the solution is then

$$F_{\text{exact}}^{\text{forced}}(x, t) = F_{\text{exact}}^{\text{resonant}}(x, t) \equiv F_k^{\text{resonant}} \, \mathrm{e}^{i(kx+\Omega_k t)}, \quad \beta = \alpha + kU + \Omega_k = 0, \tag{2.13}$$

where

$$F_k^{\text{resonant}} \equiv R_k t. \tag{2.14}$$

Equation (2.13) corresponds to the initial condition

$$F_{\text{exact}}^{\text{resonant}}(x, 0) = 0, \quad \beta = \alpha + kU + \Omega_k = 0. \tag{2.15}$$

(iv) *Complete solution.* For each Fourier component, the complete solution is the sum of the free solution and the forced solution, see (2.7). For a single forcing frequency, Ω_k, the solution is therefore either

$$F_{\text{exact}}(x, t) = F_k^{\text{free}} \, \mathrm{e}^{i\{kx-(\alpha+kU-i\beta)t\}} + \frac{R_k \, \mathrm{e}^{i(kx+\Omega_k t)}}{\beta + i(\alpha + kU + \Omega_k)}, \quad \text{for } \beta + i(\alpha + kU + \Omega_k) \neq 0, \tag{2.16}$$

or

$$F_{\text{exact}}(x, t) = (F_k^{\text{free}} + R_k t) \, \mathrm{e}^{i\{kx-(\alpha+kU)t\}}, \quad \text{for } \beta = \alpha + kU + \Omega_k = 0. \tag{2.17}$$

In general, though, each wave number may be forced by a spectrum of frequencies. The single forcing term in (2.4), $R_k \exp\{i(kx + \Omega_k t)\}$, is then replaced by

$$\sum_{\Omega_k} R_{k,\Omega_k} \, \mathrm{e}^{i(kx+\Omega_k t)}, \tag{2.18}$$

where the sum is over all frequencies, Ω_k, and R_{k,Ω_k} is the amplitude of the forcing at frequency Ω_k. Then, by linear superposition, the complete solution is either

$$F_{\text{exact}}(x, t) = F_k^{\text{free}} \, \mathrm{e}^{i\{kx-(\alpha+kU-i\beta)t\}} + \sum_{\Omega_k} \frac{R_{k,\Omega_k} \, \mathrm{e}^{i(kx+\Omega_k t)}}{\beta + i(\alpha + kU + \Omega_k)} \quad \text{for } \beta \neq 0, \tag{2.19}$$

or

$$F_{\rm exact}(x,t) = (F_k^{\rm free} + R_{k,-(\alpha+kU)}t)\,{\rm e}^{i\{kx-(\alpha+kU)t\}} + \sum_{\Omega_k \neq -(\alpha+kU)} \frac{R_{k,\Omega_k}\,{\rm e}^{i(kx+\Omega_k t)}}{i(\alpha+kU+\Omega_k)} \quad \text{for } \beta = 0. \tag{2.20}$$

(v) *Forced steady-state* ($\Omega_k = 0$) *solution for* $\beta + i(\alpha + kU) \neq 0$. An interesting special case is the forced steady-state response of the solution when there is no resonance. Since then $\beta + i(\alpha + kU) \neq 0$, it can be seen from (2.16) that a steady-state solution

$$F_{\rm exact}^{\rm steady}(x) = \frac{R_k\,{\rm e}^{ikx}}{\beta + i(\alpha + kU)} \tag{2.21}$$

to (2.4) exists when

$$\Omega_k \equiv 0, \tag{2.22}$$

and either

$$F_k^{\rm free} \equiv 0, \tag{2.23}$$

or

$$\beta > 0. \tag{2.24}$$

The first and second conditions respectively exorcize the transient forced and free components of the flow, while the third ensures that any initially non-zero transient ultimately decays to zero, albeit after infinite time.

3. Applications of the canonical problem to some coupling discretizations

The canonical problem proposed above is now applied to analyse some physics–dynamics coupling strategies. In what follows, the left-hand side of (2.4), which represents the dynamics, is always discretized using an off-centred, semi-implicit, semi-Lagrangian scheme. However, four different discretizations are applied to the right-hand-side terms and each represents a different coupling strategy.

(*a*) *'Implicit' ('explicit')*

The 'implicit' ('explicit') discrete forms of (2.4) are obtained by applying an off-centred semi-implicit semi-Lagrangian discretization (Rivest *et al.* (1994)) to all terms in (2.4), giving the discrete form as

$$\begin{aligned}\frac{F(x,t+\Delta t) - F(x-a,t)}{\Delta t} &+ i\alpha\{\xi_1 F(x,t+\Delta t) + (1-\xi_1)F(x-a,t)\} \\ &= -\beta\{\xi_2 F(x,t+\Delta t) + (1-\xi_2)F(x-a,t)\} + \xi_3 R_k\,{\rm e}^{ikx+i\Omega_k(t+\Delta t)} \\ &\quad + (1-\xi_3)R_k\,{\rm e}^{ik(x-a)+i\Omega_k t},\end{aligned} \tag{3.1}$$

where

$$a = U\Delta t, \tag{3.2}$$

is the displacement in x of a particle during time Δt so that $x - a$ corresponds to the semi-Lagrangian departure point. The off-centring coefficients ξ_1, ξ_2 and ξ_3, possibly of different value ($\xi_1 = \xi_2 = \xi_3 = ½$ corresponds to the classical Crank–Nicolson scheme), have been introduced in anticipation of analysis of resonance.

If $\xi_2 \neq 0$, then (3.1) is an implicit coupling between the dynamics (left-hand-side) and physics (right-hand-side) terms, since $F(x, t + \Delta t)$ appears in both. If, however, $\xi_2 = 0$, then it is an explicit coupling (cf. Eq. 9 of CLZ98 for the special case $\xi_1 = \xi_3 = ½$, $\beta = 0$ and $\Omega_k = 0$).

As noted above, the $R_k \exp\{i(kx + \Omega_k t)\}$ term can be interpreted in different ways, and here are two. Setting $\Omega_k = 0$, it can represent stationary forcing (such as that caused by orography) and the usual averaging of dynamical terms along the trajectory would set ξ_3 identically equal to ξ_1. Setting $\xi_3 = 0$, it can instead represent a spatio-temporal physics forcing which is explicitly evaluated upstream in the spirit of CLZ98.

In the context of the enhanced canonical problem, the explicit and implicit couplings can be considered to represent two ideals. On the one hand, the explicit coupling is ideal from the point of view of simplicity. The physics terms, which in a full model are highly nonlinear, are evaluated explicitly using values at time t only, but the time step is consequently, and unfortunately, severely limited by the stability constraint associated with the explicit time-discretization of the fast parametrizations (e.g. vertical diffusion). On the other hand, the implicit coupling is then arguably ideal because of its good stability and accuracy properties but, in the full model context, the coupling of the physics with the dynamics then leads to a highly nonlinear discrete problem to solve.

(*b*) *'Split-implicit'*

Discretizing the terms on the right-hand side of (2.4) in the spirit of the 'split-implicit' coupling of CLZ98 results in the two-step discretization of (2.4)

$$\frac{F^*(x) - F(x-a, t)}{\Delta t} + i\alpha\{\xi_1 F^*(x) + (1-\xi_1)F(x-a, t)\} = 0, \tag{3.3}$$

$$\begin{aligned}\frac{F(x, t+\Delta t) - F^*(x)}{\Delta t}& \\ = -\beta F(x, t+\Delta t) &+ \xi_3 R_k\, \mathrm{e}^{ikx + i\Omega_k(t+\Delta t)} + (1-\xi_3)R_k\, \mathrm{e}^{ik(x-a)+i\Omega_k t},\end{aligned} \tag{3.4}$$

where again the off-centring coefficients, possibly of different value ($\xi_1 = ½$ corresponds to the classical centred semi-implicit scheme for the dynamics terms), have been introduced in anticipation of analysis of resonance. Both steps are implicit and the second step, in the context of vertical diffusion, can be accomplished by solving a set of tri-diagonal problems in the vertical. The off-centring of the time-dependent forcing term allows the analysis to include the full range of time weightings of this term from fully implicit to fully explicit. The first step of the split-implicit coupling is a dynamics-only predictor, while the second is a physics-only corrector. This keeps the physics discretization distinct from that of the dynamics. Eliminating F^* from (3.3) and (3.4) yields the equivalent coupling equation

$$\begin{aligned}\frac{F(x, t+\Delta t) - F(x-a, t)}{\Delta t} &+ i\alpha\{\xi_1(1+\beta\Delta t)F(x, t+\Delta t) + (1-\xi_1)F(x-a, t)\} \\ = -\beta F(x, t+\Delta t) &+ (1 + i\alpha\Delta t\xi_1)\{\xi_3 R_k\, \mathrm{e}^{ikx+i\Omega_k(t+\Delta t)} \\ &+ (1-\xi_3)R_k\, \mathrm{e}^{ik(x-a)+i\Omega_k t}\}.\end{aligned} \tag{3.5}$$

Instead of evaluating the forcing term $\xi_3 R_k \exp\{ikx + i\Omega_k(t+\Delta t)\} + (1-\xi_3)R_k \times \exp\{ik(x-a) + i\Omega_k t\}$ in the physics sub-step (3.4), it could alternatively be evaluated in the dynamics sub-step (3.3) where, for example, it could represent stationary orographic forcing (with Ω_k set identically to zero). The $(1 + i\alpha\Delta t\xi_1)$ factor on the right-hand side of the equivalent coupling equation (3.5) would then be absent. Since the left-hand side of this equation would remain unchanged, the only impact this has on the analysis is to modify the spatial averaging of the forcing term slightly, with the consequences that the conditions for the stability and for resonance to occur would also remain unchanged, and that only the amplitude of the forced response would be slightly modified. Thus this alternative split-implicit coupling is not pursued further herein.

(*c*) *'Symmetrized split-implicit'*

The implicit coupling is $O(\Delta t^2)$ accurate if $\xi_1 = \xi_2 = \xi_3 = ½$ (or almost so with slight decentring), it is unconditionally stable, and it leads to the exact steady-state for constant forcing. Although from the stability and accuracy viewpoints the implicit coupling is very good, it nevertheless has the important drawback of being difficult to implement in a computationally efficient manner. This motivates the 'symmetrized split-implicit' coupling which aims to achieve the advantages of the implicit coupling without compromising computational efficiency. It comprises the following three-step discretization of (2.4)

$$\frac{F^*(x) - F(x,t)}{\Delta t} = -(1-\xi_2)\beta F(x,t) + (1-\xi_3)R_k\,\mathrm{e}^{ikx+i\Omega_k t}, \tag{3.6}$$

$$\frac{F^{**}(x) - F^*(x-a)}{\Delta t} + i\alpha\{\xi_1 F^{**}(x) + (1-\xi_1)F^*(x-a)\} = 0, \tag{3.7}$$

$$\frac{F(x,t+\Delta t) - F^{**}(x)}{\Delta t} = -\xi_2\beta F(x,t+\Delta t) + \xi_3 R_k\,\mathrm{e}^{ikx+i\Omega_k(t+\Delta t)}, \tag{3.8}$$

where off-centring parameters ($\xi_1 = ½$ corresponds to the classical centred semi-implicit semi-Lagrangian scheme for the dynamics terms) have again been introduced in anticipation of analysis of resonance. Note that $F^*(x-a)$ in (3.7) satisfies (3.6) but with x everywhere replaced by $(x-a)$. The first step is explicit and the other two are implicit. The third step, in the context of vertical diffusion, can again be accomplished by solving a set of tri-diagonal problems in the vertical.

The symmetrized split-implicit coupling can be considered to be a 'weak coupling', and comprises two physics discretizations symmetrically arranged around the dynamics discretization. Eliminating F^* and F^{**} from (3.6) to (3.8) yields the equivalent coupling equation

$$\begin{aligned}
&\frac{F(x,t+\Delta t) - F(x-a,t)}{\Delta t} + i\alpha[\xi_1(1+\beta\Delta t\xi_2)F(x,t+\Delta t) \\
&\qquad + (1-\xi_1)\{1-\beta\Delta t(1-\xi_2)\}F(x-a,t)] \\
&\quad = -\beta\{\xi_2 F(x,t+\Delta t) + (1-\xi_2)F(x-a,t)\} + (1+i\alpha\Delta t\xi_1)\xi_3 R_k\,\mathrm{e}^{ikx+i\Omega_k(t+\Delta t)} \\
&\qquad + \{1-i\alpha\Delta t(1-\xi_1)\}(1-\xi_3)R_k\,\mathrm{e}^{ik(x-a)+i\Omega_k t}.
\end{aligned} \tag{3.9}$$

Instead of evaluating the forcing terms $(1-\xi_3)R_k\exp\{ik(x-a)+i\Omega_k t\}$ and $\xi_3 R_k\exp\{ikx+i\Omega_k(t+\Delta t)\}$ on the respective right-hand sides of the physics sub-steps (3.6) and (3.8), they could instead be evaluated together on the right-hand side of (3.7) as $(1-\xi_3)R_k\exp\{ik(x-a)+i\Omega_k t\} + \xi_3 R_k\exp\{ikx+i\Omega_k(t+\Delta t)\}$, and again it would be possible to represent stationary orographic forcing in the dynamics sub-step. The $(1+i\alpha\Delta t\xi_1)$ and $\{1-i\alpha\Delta t(1-\xi_1)\}$ factors on the right-hand side of the equivalent coupling equation (3.9) would then be absent. Since the left-hand side of this equation would remain unchanged, the only impact this has on the analysis is to modify the spatial averaging of the forcing term slightly, with the consequences that the conditions for stability and for resonance to occur would also remain unchanged, and that only the amplitude of the forced response would be slightly modified. Thus, this alternative symmetrized split-implicit coupling is not pursued further herein.

4. Stability and accuracy of the free solution

(*a*) *Stability*

Stability of the free solution is examined via a standard von Neumann stability analysis (Haltiner and Williams 1980).

For the implicit coupling (3.1), the free component of the flow satisfies

$$\frac{F^{\text{free}}(x, t+\Delta t) - F^{\text{free}}(x-a, t)}{\Delta t} + i\alpha\{\xi_1 F^{\text{free}}(x, t+\Delta t) + (1-\xi_1) F^{\text{free}}(x-a, t)\}$$
$$+\beta\{\xi_2 F^{\text{free}}(x, t+\Delta t) + (1-\xi_2) F^{\text{free}}(x-a, t)\} = 0. \tag{4.1}$$

From the initial condition (2.9), $F^{\text{free}}(x, t)$ is expanded as

$$F^{\text{free}}(x, t) = F_k^{\text{free}}\, \mathrm{e}^{i(kx+\omega t)}. \tag{4.2}$$

Noting that here $a \equiv U\Delta t$ (though for general, spatially varying U a trajectory equation would need to be solved for a), this gives

$$E = \frac{1 - \beta\Delta t(1-\xi_2) - i\alpha\Delta t(1-\xi_1)}{1 + \beta\Delta t\xi_2 + i\alpha\Delta t\xi_1}, \tag{4.3}$$

where

$$E \equiv \frac{F^{\text{free}}(x, t+\Delta t)}{F^{\text{free}}(x, t)} = \mathrm{e}^{i(\omega+kU)\Delta t}. \tag{4.4}$$

Given that α and $\beta \geqslant 0$ are both real, and that $|E| \leqslant 1$ for stability, this yields the conditional stability condition

$$-\beta\Delta t\{1 + \beta\Delta t(\xi_2 - ½)\} \leqslant \alpha^2\Delta t^2(\xi_1 - ½). \tag{4.5}$$

For the implicit case (for which $\xi_2 \neq 0$), (4.5) is easy to satisfy *robustly*, i.e. independently of the values of $\alpha\Delta t$ and $\beta\Delta t \geqslant 0$, by simply requiring the left (right)-hand sides to be less (greater) than or equal to zero, thus leading to

$$\xi_1, \xi_2 \geqslant ½. \tag{4.6}$$

For the explicit case (where $\xi_2 \equiv 0$), (4.5) can be very restrictive, depending upon the values of the parameters. If β is strictly greater than zero but small compared to $|\alpha|$, and $\xi_1 < ½$, then (4.5) asymptotically reduces to

$$\Delta t \leqslant \frac{\beta}{\alpha^2(½ - \xi_1)}, \tag{4.7}$$

thus negating much or all of the time-step advantage of the semi-implicit scheme. This serious drawback is easily avoided by respecting the condition

$$\xi_1 \geqslant ½. \tag{4.8}$$

However, there remains another important asymptotic limit, viz. when α^2 is negligibly small, and then (4.5) reduces to

$$0 \leqslant \beta\Delta t \leqslant 2. \tag{4.9}$$

TABLE 1. DISPERSION RELATION (E) AND STABILITY CRITERION AS A FUNCTION OF COUPLING SCHEME

Coupling scheme	Dispersion relation (E)	Robust stability condition(s)
Exact	$\exp\{-(i\alpha+\beta)\Delta t\}$	$0 \leqslant \beta\Delta t$
Explicit ($\xi_2 \equiv 0$)	$\dfrac{1-\beta\Delta t - i\alpha\Delta t(1-\xi_1)}{1+i\alpha\Delta t\xi_1}$	$0 \leqslant \beta\Delta t \leqslant 2$
Implicit	$\dfrac{1-\beta\Delta t(1-\xi_2) - i\alpha\Delta t(1-\xi_1)}{1+\beta\Delta t\xi_2 + i\alpha\Delta t\xi_1}$	$0 \leqslant \beta\Delta t$ and $\xi_1, \xi_2 \geqslant ½$
Split-implicit	$\left(\dfrac{1}{1+\beta\Delta t}\right)\left\{\dfrac{1-i\alpha\Delta t(1-\xi_1)}{(1+i\alpha\Delta t\xi_1)}\right\}$	$0 \leqslant \beta\Delta t$ and $\xi_1 \geqslant ½$
Symmetrized split-implicit	$\left\{\dfrac{1-i\alpha\Delta t(1-\xi_1)}{1+i\alpha\Delta t\xi_1}\right\}\left\{\dfrac{1-\beta\Delta t(1-\xi_2)}{1+\beta\Delta t\xi_2}\right\}$	$0 \leqslant \beta\Delta t$ and $\xi_1, \xi_2 \geqslant ½$

This condition is very restrictive for fast damping processes, such as vertical diffusion in the boundary layer at high resolution, thus rendering the explicit coupling computationally inefficient for practical applications. Indeed, although satisfaction of (4.9) guarantees stability of the free solution, the solution does not necessarily respect physical fidelity. This is particularly easy to see in the special case where $\alpha\Delta t$ is vanishingly small and $\beta\Delta t = 2$: (4.3) then reduces to $E = -1$ and $F^{\text{free}}(t)$ spuriously changes sign on alternate time-steps with no damping whatsoever. To avoid this situation and ensure that the damping rate per time step, $|E|$, decreases monotonically as the damping coefficient, β, increases, it is preferable to satisfy the more restrictive condition

$$0 \leqslant \beta\Delta t \leqslant 1. \tag{4.10}$$

The dispersion relations and associated stability conditions for the other coupling schemes of section 3 may be found in a similar manner to that given above for the implicit and explicit couplings, and the ensuing results are summarized in Table 1. While the explicit coupling scheme for the revised canonical problem has a very restrictive stability condition, the other three share the advantage of being stable independently of $\alpha\Delta t$ and $\beta\Delta t$ provided the time-weighting parameters are chosen appropriately.

(*b*) *Accuracy*

Applying (4.4) to the exact free solution (2.8) leads to

$$\begin{aligned} E_{\text{exact}} &\equiv \exp\{i(\omega_{\text{exact}} + kU)\Delta t\} = \exp\{-(i\alpha+\beta)\Delta t\} \\ &= 1 - (i\alpha+\beta)\Delta t + \frac{(i\alpha+\beta)^2\Delta t^2}{2} - \frac{(i\alpha+\beta)^3\Delta t^3}{6} + O(\Delta t)^4, \end{aligned} \tag{4.11}$$

where

$$\omega_{\text{exact}} = -\alpha - kU + i\beta, \tag{4.12}$$

is the exact value of ω.

For the implicit coupling, by expanding (4.3) in terms of Δt, the corresponding discrete expression is

$$\begin{aligned} E_{\text{implicit}} = 1 &- (i\alpha+\beta)\Delta t + (i\alpha+\beta)(i\alpha\xi_1 + \beta\xi_2)\Delta t^2 \\ &-(i\alpha+\beta)(i\alpha\xi_1+\beta\xi_2)^2\Delta t^3 + O(\Delta t)^4. \end{aligned} \tag{4.13}$$

TABLE 2. $E - E_{\text{exact}}$ AS A FUNCTION OF COUPLING SCHEME

Coupling scheme	$E - E_{\text{exact}}$
Explicit	$\left\{i\alpha\left(\xi_1 - \frac{1}{2}\right) - \frac{\beta}{2}\right\}(i\alpha + \beta)\Delta t^2 + O(\Delta t)^3$
Implicit	$\left\{i\alpha\left(\xi_1 - \frac{1}{2}\right) + \beta\left(\xi_2 - \frac{1}{2}\right)\right\}(i\alpha + \beta)\Delta t^2 - \left\{(i\alpha\xi_1 + \beta\xi_2)^2 - \frac{(i\alpha + \beta)^2}{6}\right\}(i\alpha + \beta)\Delta t^3 + O(\Delta t)^4$
Split-implicit	$\left\{\left(\xi_1 - \frac{1}{2}\right)(i\alpha)^2 + \frac{\beta^2}{2}\right\}\Delta t^2 + O(\Delta t)^3$
Symmetrized split-implicit	$\{(i\alpha)^2(\xi_1 - \frac{1}{2}) + \beta^2(\xi_2 - \frac{1}{2})\}\Delta t^2 - \{(i\alpha)^3(\xi_1^2 - \frac{1}{6}) + (i\alpha)^2\beta(\xi_1 - \frac{1}{2}) + i\alpha\beta^2(\xi_2 - \frac{1}{2}) + \beta^3(\xi_2^2 - \frac{1}{6})\}\Delta t^3 + O(\Delta t)^4$

Thus, for a centred scheme ($\xi_1 = \xi_2 = ½$), the discrete transient amplitude (4.13) agrees with the exact one (4.11) to $O(\Delta t)^2$ even when $\beta \neq 0$, i.e. in the presence of damping. Off-centring for the implicit coupling, however, reduces the accuracy to $O(\Delta t)$, but negligibly so for negligibly small off-centring. The accuracy of the discrete transient amplitude ($E - E_{\text{exact}}$) for the other three coupling schemes may be found in a similar manner, and the ensuing results are summarized in Table 2.

Examination of Table 2 shows that all couplings have the same formal order of accuracy, with the discrete transient amplitude agreeing with the exact one to $O(\Delta t)$. However, for centred schemes (for which $\xi_1 = \xi_2 = ½$) and irrespective of the values of α and β, the implicit and symmetrized split-implicit couplings are more accurate, agreeing with the exact transient amplitude to $O(\Delta t)^2$. But centring the explicit and split-implicit couplings does not increase the formal order of accuracy except for the special case $\beta = 0$, i.e. in the absence of damping. Off-centring formally reduces the accuracy of the implicit and symmetrized split-implicit couplings, but negligibly so for small off-centring.

5. The forced non-resonant response

(*a*) *Forced response*

By analogy with the continuous case, the complete solution of (3.1) is the sum of the complementary function and a particular integral. The complementary function is the general solution of the homogeneous equation, i.e. the equation obtained by setting the forcing coefficient R_k identically to zero on the right-hand side of (3.1). This is referred to above as the free solution, and it is given by (4.2), where ω is determined from (4.3) and (4.4). The particular integral is any solution of the complete inhomogeneous equation and includes all forcings.

For the implicit coupling, a simple way of obtaining a particular integral of (3.1) is to seek solutions of the form

$$F^{\text{forced}}(x, t) = F_k^{\text{forced}}\, e^{i(kx + \Omega_k t)}, \tag{5.1}$$

TABLE 3. FORCED NON-RESONANT RESPONSE $(F_k^{\text{forced}}/R_k)$ AS A FUNCTION OF COUPLING SCHEME

Coupling scheme	F_k^{forced}/R_k
Exact	$\dfrac{1}{\beta + i(\alpha + kU + \Omega_k)}$
Explicit	$\dfrac{\{\xi_3\mathcal{E} + (1-\xi_3)\}\Delta t}{(1 + i\alpha\Delta t\xi_1)(\mathcal{E} - E_{\text{explicit}})}$
Implicit	$\dfrac{\{\xi_3\mathcal{E} + (1-\xi_3)\}\Delta t}{(1 + i\alpha\Delta t\xi_1 + \beta\Delta t\xi_2)(\mathcal{E} - E_{\text{implicit}})}$
Split-implicit	$\dfrac{\{\xi_3\mathcal{E} + (1-\xi_3)\}\Delta t}{(1 + \beta\Delta t)(\mathcal{E} - E_{\text{split-implicit}})}$
Symmetrized split-implicit	$\dfrac{[(1 + i\alpha\Delta t\xi_1)\xi_3\mathcal{E} + \{1 - i\alpha\Delta t(1-\xi_1)\}(1-\xi_3)]\Delta t}{(1 + i\alpha\Delta t\xi_1)(1 + \beta\Delta t\xi_2)(\mathcal{E} - E_{\text{ssi}})}$

TABLE 4. $\{\beta + i(\alpha + kU + \Omega_k)\}^2\{F_k^{\text{forced}}/R_k - (F_k^{\text{forced}}/R_k)_{\text{exact}}\}$ AS A FUNCTION OF COUPLING SCHEME

Coupling scheme	$\{\beta + i(\alpha + kU + \Omega_k)\}^2\{F_k^{\text{forced}}/R_k - (F_k^{\text{forced}}/R_k)_{\text{exact}}\}$
Explicit	$\{(\xi_1 - \xi_3)\alpha + i\xi_3\beta - (\xi_3 - ½)(kU + \Omega_k)\}(kU + \Omega_k)\Delta t + O(\Delta t^2)$
Implicit	$\{(\xi_1 - \xi_3)\alpha + i(\xi_3 - \xi_2)\beta - (\xi_3 - ½)(kU + \Omega_k)\}(kU + \Omega_k)\Delta t + O(\Delta t^2)$
Split implicit	$[-\{\xi_1\alpha + \xi_3(kU + \Omega_k)\}\alpha + \{i(\xi_3 - 1)\beta - (\xi_3 - ½)(kU + \Omega_k)\}(kU + \Omega_k)]\Delta t + O(\Delta t^2)$
Symmetrized split-implicit	$-[\{(\xi_1 + \xi_3 - 1)\alpha + i(\xi_2 - \xi_3)\beta + 2(\xi_3 - ½)(kU + \Omega_k)\}\alpha + \{i(\xi_3 - \xi_2)\beta - (\xi_3 - ½)(kU + \Omega_k)\}(kU + \Omega_k)]\Delta t + O(\Delta t^2)$

and to then determine F_k^{forced}. Thus

$$(F_k^{\text{forced}})_{\text{implicit}} = \left[\frac{\{\xi_3\mathcal{E} + (1-\xi_3)\}\Delta t}{(1 + i\alpha\Delta t\xi_1 + \beta\Delta t\xi_2)\mathcal{E} - \{1 - i\alpha\Delta t(1-\xi_1) - \beta\Delta t(1-\xi_2)\}}\right] R_k$$
$$= \left[\frac{\{\xi_3\mathcal{E} + (1-\xi_3)\}\Delta t}{(1 + i\alpha\Delta t\xi_1 + \beta\Delta t\xi_2)(\mathcal{E} - E_{\text{implicit}})}\right] R_k, \qquad (5.2)$$

where

$$\mathcal{E} = \mathrm{e}^{i(\Omega_k + kU)\Delta t}, \qquad (5.3)$$

$\alpha\Delta t$ and $\beta\Delta t$ are non-dimensional parameters, E_{implicit} is given by (4.3), and (5.1) and (5.2) are valid provided the denominator in (5.2) is non-zero. The singular (resonant) case is analysed in section 6. The forced non-resonant response for the other three coupling schemes of section 3 may be found in a similar manner and results for F_k^{forced}/R_k are summarized in Table 3, where $E_{\text{coupling scheme}}$ is defined by the middle column of Table 1.

The difference of the discrete forced non-resonant response F_k^{forced}/R_k from the exact one, (2.10) and (2.11), normalized for convenience by $\{\beta + i(\alpha + kU + \Omega_k)\}^{-2}$, has been derived from Table 3 and is displayed in Table 4 as a function of coupling scheme. Examination of this table shows that all couplings have the same formal order of accuracy, with the discrete amplitude agreeing with the exact one to $O(\Delta t)$. However, for centred schemes (for which $\xi_j = ½$) and irrespective of the values of α and β, the implicit and symmetrized split-implicit couplings are more accurate and then agree with the exact result to $O(\Delta t^2)$. But centring the explicit and split-implicit couplings does not increase the formal order of accuracy when $\beta > 0$, i.e. in the presence of damping. Even when $\beta = 0$, centring the split-implicit scheme still does not make it second-order accurate. Off-centring formally reduces the accuracy of the implicit and symmetrized split-implicit couplings, but negligibly so for small off-centring.

(*b*) *Steady-state response*

If $k \equiv 0$ and $\Omega_k \equiv 0$, i.e. the forcing is constant (as in the simpler canonical problem of CLZ98), then, firstly, $\mathcal{E} \equiv 1$, secondly, a forced steady-state response exists provided either (2.23) or (2.24) holds, and, thirdly, (5.1) reduces to the exact result (2.21) (with $k = 0$) regardless of the precise values of the off-centring parameters ξ_1, ξ_2 and ξ_3. These results also hold for the explicit coupling since it is the special case of the implicit coupling where $\xi_2 \equiv 0$. For the split-implicit coupling-scheme, however, this is not the case, nor is it for the symmetrized split-implicit coupling-scheme unless the scheme is centred.

For the split-implicit scheme, substitution of $k \equiv 0$, $\Omega_k \equiv 0$, $\mathcal{E} \equiv 1$ and for $E_{\text{split-implicit}}$ from Table 1 into the split-implicit expression given in Table 3 yields

$$(F_0^{\text{forced}})_{\text{split-implicit}} = \frac{R_0}{i\alpha/(1 + i\alpha\Delta t\xi_1) + \beta}. \tag{5.4}$$

Thus for the split-implicit coupling, consistent with the corresponding situation for the simpler canonical problem examined by CLZ98 (where $\beta \equiv 0$), the constant forced component of the flow (5.4) no longer corresponds to the exact result (2.21). Consider now the ratio

$$\left|\frac{(F_0^{\text{forced}})_{\text{split-implicit}}}{(F_0^{\text{forced}})_{\text{exact}}}\right|^2 = \frac{(1 + \alpha^2\Delta t^2\xi_1^2)\{(\alpha\Delta t)^2 + (\beta\Delta t)^2\}}{(\alpha\Delta t)^2(1 + \beta\Delta t\xi_1)^2 + (\beta\Delta t)^2}, \tag{5.5}$$

i.e. the squared amplitude of the ratio of approximate to exact results. From (5.5) it can be seen that $|(F_0^{\text{forced}})_{\text{split-implicit}}/(F_0^{\text{forced}})_{\text{exact}}|$ is less than or greater than unity according to whether $\beta\Delta t$ is greater or less than the critical value $\xi_1(\alpha\Delta t)^2/2$. As noted by CLZ98 for the special case where there is no damping (i.e. $\beta \equiv 0$), the forced response can be spuriously amplified by an order of magnitude. However, it is seen from (5.5) that a sufficiently large damping can inhibit the spurious amplification of the constant forced component of the flow or, even, spuriously diminish such forcing. The impact of off-centring the time-discretization of the semi-implicit terms is to increase the threshold at which strong damping diminishes the amplitude of the constant forced component.

For the symmetrized split-implicit scheme, substitution of $k \equiv 0$, $\Omega_k \equiv 0$, $\mathcal{E} \equiv 1$ and for E_{ssi} from Table 1 into the symmetrized split-implicit expression given in Table 3 yields

$$(F_0^{\text{forced}})_{\text{ssi}} = \frac{\{1 + i\alpha\Delta t(\xi_1 + \xi_3 - 1)\}R_0\Delta t}{i\alpha\Delta t\{1 + (\xi_1 + \xi_2 - 1)\beta\Delta t\} + \beta\Delta t}. \tag{5.6}$$

For general ξ_1, ξ_2 and ξ_3 this does not correspond to the exact result (2.21) (with $k = 0$). It does, however, do so if

$$\xi_1 + \xi_2 = \xi_1 + \xi_3 = 1, \quad \text{i.e. } \xi_2 = \xi_3 = 1 - \xi_1, \tag{5.7}$$

and these are the conditions that the symmetrized split-implicit time-weights must satisfy in order to obtain the exact steady-state for constant forcing. This, coupled with the requirement that $\xi_1, \xi_2 \geqslant ½$ for robust stability (Table 1), means that $\xi_1 = \xi_2 = ½$ is required in order to capture the exact steady-state for constant forcing, and a small decentring of the time-weights slightly perturbs the discrete steady-state away from the exact one.

6. The forced resonant response

For the implicit coupling, if the denominator of (5.2) is zero, i.e. if

$$\mathcal{E} \equiv \mathrm{e}^{i(\Omega_k + kU)\Delta t} = E_{\text{implicit}} = \frac{1 - \beta\Delta t(1-\xi_2) - i\alpha\Delta t(1-\xi_1)}{1 + \beta\Delta t\xi_2 + i\alpha\Delta t\xi_1}, \tag{6.1}$$

then the solution is singular and resonant. This provides a constraint, for resonance to occur, on the six non-dimensional parameters, ξ_1, ξ_2, $\alpha\Delta t$, $\beta\Delta t$, W and KC where

$$W \equiv \Omega_k \Delta t, \quad K \equiv k\Delta x, \quad C \equiv \frac{U\Delta t}{\Delta x}. \tag{6.2}$$

Since $(\Omega_k + kU)\Delta t$ is real, resonance is only possible if the parameters are such that the right-hand side of (6.1) lies on the unit circle in the complex plane, i.e.

$$|E_{\text{implicit}}|^2 = \frac{\{1 - \beta\Delta t(1-\xi_2)\}^2 + \alpha^2\Delta t^2(1-\xi_1)^2}{(1 + \beta\Delta t\xi_2)^2 + \alpha^2\Delta t^2\xi_1^2} = 1. \tag{6.3}$$

Satisfaction of (6.3) is a necessary but insufficient condition for resonance. It contrains the magnitude of E_{implicit} but not its phase. Resonance, real or spurious, can only occur if (6.1) is also simultaneously satisfied. Here, $W + KC \equiv (\Omega_k + kU)\Delta t$ is constrained by

$$0 \leqslant K, \ |W| \leqslant \pi, \tag{6.4}$$

in order for the forcing to be properly sampled and resolved in both space and time—any higher non-dimensionalized wavelength or forcing frequency would be aliased and indistinguishable from a wavelength or frequency satisfying these contraints, and the discretization would behave in exactly the same manner as that for the aliased wavelength or frequency.

Recall that $\beta\Delta t \geqslant 0$ and that for robust stability reasons—see text preceding (4.6)—the off-centring parameters ξ_1 and ξ_2 are constrained to be $\geqslant ½$. Inspection of (6.3) shows that for $\xi_1, \xi_2 \geqslant ½$, it can only be satisfied if $\beta\Delta t = 0$ and additionally either $\alpha\Delta t = 0$ or $\xi_1 = ½$. Consequently, resonance can only occur if $\beta\Delta t = 0$ and additionally either $\alpha\Delta t = 0$ or $\xi_1 = ½$, and it is suppressed if $\xi_1 > ½$ and $\alpha\Delta t \neq 0$.

Similarly, necessary resonance conditions for the other coupling schemes of section 3 may be found by setting the denominators in Table 3 to zero. They all have the form of (6.3), i.e. $|E_{\text{coupling scheme}}|^2 = 1$, and $|E_{\text{coupling scheme}}|^2$ is displayed in Table 5 as a function of coupling scheme.

Setting $|E_{\text{coupling scheme}}|^2$ of Table 5 to unity (a necessary condition for resonance to occur), it can be shown that all coupling schemes lead to the same necessary conditions for resonance as the implicit coupling. Thus, in summary, for all four schemes there are only two possible ways for resonance to occur: (a) if $\alpha\Delta t = \beta\Delta t = 0$ and (b) if $\beta\Delta t = 0$ and $\xi_1 = ½$. For all four schemes it turns out that inserting these conditions

TABLE 5. $|E|^2$ AS A FUNCTION OF COUPLING SCHEME

Coupling scheme	$\lvert E_{\text{coupling scheme}}\rvert^2$
Explicit	$\dfrac{(1-\beta\Delta t)^2+\alpha^2\Delta t^2(1-\xi_1)^2}{1+\alpha^2\Delta t^2\xi_1^2}$
Implicit	$\dfrac{\{1-\beta\Delta t(1-\xi_2)\}^2+\alpha^2\Delta t^2(1-\xi_1)^2}{(1+\beta\Delta t\xi_2)^2+\alpha^2\Delta t^2\xi_1^2}$
Split-implicit	$\dfrac{1+\alpha^2\Delta t^2(1-\xi_1)^2}{(1+\beta\Delta t)^2(1+\alpha^2\Delta t^2\xi_1^2)}$
Symmetrized split-implicit	$\left\{\dfrac{1-\beta\Delta t(1-\xi_2)}{1+\beta\Delta t\xi_2}\right\}^2\left\{\dfrac{1+\alpha^2\Delta t^2(1-\xi_1)^2}{1+\alpha^2\Delta t^2\xi_1^2}\right\}$

into the appropriate analogues of (6.1) leads to the same expressions for $\mathcal{E}$, viz.

$$\mathcal{E} \equiv \mathrm{e}^{i(W+KC)} = 1, \tag{6.5}$$

for case (a), and

$$\mathcal{E} \equiv \mathrm{e}^{i(W+KC)} = \frac{1-i\alpha\Delta t/2}{1+i\alpha\Delta t/2}, \tag{6.6}$$

for case (b). These two cases are now examined in detail.

(i) *Case (a),* $\alpha\Delta t=\beta\Delta t=0$. From (6.5) resonance occurs, independently of the precise value of the off-centring parameters ξ_1 and ξ_2, when

$$W+KC \equiv (\Omega_k+kU)\Delta t = 2n\pi, \quad n=0,\pm1,\pm2,\ldots. \tag{6.7}$$

When $n=0$ this agrees with the exact condition (2.13) for physical resonance. However, spurious resonance occurs when $n=\pm1,\pm2,\ldots$.

Since, from (6.7),

$$\max|W,\,K| \equiv \max|\Omega_k\Delta t,\,k\Delta x| \leqslant \pi, \tag{6.8}$$

these spurious resonances can be avoided by taking the time step such that

$$C \equiv \frac{U\Delta t}{\Delta x} < 1. \tag{6.9}$$

From (6.7), for stationary forcing (i.e. $W\equiv 0$), $C<2$ is sufficient to suppress them. Thus (6.9), for time-dependent forcing, is *twice as restrictive* a condition as the corresponding one for stationary forcing.

These spurious resonances can alternatively be removed by damping, and this can be either explicit damping with a strictly-positive value of β, or implicit damping due to interpolation (see section 7). They cannot, however, be removed by off-centring since (6.7) is insensitive to any variation in the off-centring parameters.

(ii) *Case (b),* $\xi_1=\frac{1}{2}$, $\beta\Delta t=0$. Solving (6.6) for $(W+KC)$ and simplifying, gives the result that resonance occurs if

$$\cos(W+KC)=\frac{1-(\alpha\Delta t/2)^2}{1+(\alpha\Delta t/2)^2}, \quad \text{and} \quad \sin(W+KC)=\frac{-2(\alpha\Delta t/2)}{1+(\alpha\Delta t/2)^2}. \tag{6.10}$$

These conditions can be combined to give the necessary requirement

$$\tan\left\{\frac{(\Omega_k + kU)\Delta t}{2}\right\} \equiv \tan\left(\frac{W+KC}{2}\right) = \frac{-\alpha\Delta t}{2} = -\left(\frac{\alpha}{\Omega_k + kU}\right)\left(\frac{W+KC}{2}\right), \tag{6.11}$$

but care must then be exercised to select valid solutions. This is because not all solutions of (6.11) are necessarily solutions of (6.10). (The usual loss of phase information when condensing the real and imaginary parts of the polar representation into a single expression for the tangent of the angle introduces spurious solutions.) Consequently, (6.10) must normally be used to select the proper quadrant in the complex plane and remove any ambiguity. This is not, however, necessary in the present context since α and $\{1-(\alpha\Delta t/2)^2\}$ can take either sign, and therefore, from (6.10), $\exp\{i(W+KC)\}$ can be in any quadrant.

If $W=-KC$, then $\alpha=0$ from (6.10) and this solution corresponds to a special case of physical resonance (cf. (2.13)). If, however, $(W+KC)\neq 0$, graphical consideration of (6.11), i.e. plotting the left- and right-hand sides as functions of $(W+KC)/2$, shows that other resonant solutions exist. Noting that α can take either sign and that the slope of $\tan x$ at the origin is unity, we see there are two regimes.

(ii.a) $0\leqslant|(W+KC)/2|<\pi/2$ *(physical resonance).* For $|\alpha/(\Omega_k+kU)|>1$ (see Fig. 1) and bearing in mind that α can be of either sign, there are two resonances, with each associated α being of opposite sign to Ω_k+kU. Expanding (6.11) for small Δt shows that these solutions correspond to physical resonance since they then approximate the exact result $\alpha+kU+\Omega_k=0$, $\beta=0$, cf. (2.13). As Δt is increased, the two resonances still approximate the physical ones, albeit with diminishing accuracy, and the discrete resonant forcing frequency is larger in magnitude than the analytic one. For the limiting case $|\alpha|=(\Omega_k+kU)$ (see limiting line of Fig. 1), where $y=\tan\{(W+KC)/2\}$ is tangent to $y=|\alpha/(\Omega_k+kU)|\{(W+KC)/2\}$ at the origin, the two physical resonances coalesce into the single physical resonance when $\alpha=-(kU+\Omega_k)=0$. For $|\alpha/(\Omega_k+kU)|<1$ (see Fig. 2), no resonance occurs for $0<|(W+KC)/2|<\pi/2$.

(ii.b) $|(W+KC)/2|>\pi/2$ *(spurious resonance).* In this regime (see Figs. 1, 2), noting that α can be of either sign, resonances occur when

$$\frac{(\Omega_k+kU)\Delta t}{2} \equiv \frac{(W+KC)}{2} \approx \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots, \tag{6.12}$$

where K and W are constrained to satisfy (6.8). These resonances are all spurious. For a stationary orographic forcing ($\Rightarrow W=0$), these spurious resonances can be avoided, as noted by Rivest *et al.* (1994), by taking a sufficiently small time-step such that

$$C \equiv \frac{U\Delta t}{\Delta x} < 1. \tag{6.13}$$

For time-dependent forcings such that $W\equiv\Omega_k\Delta t\neq 0, \pi$, they can still in principle be avoided by taking a sufficiently-small time-step. However, this time step becomes very small indeed as W approaches π and becomes excessively restrictive. These spurious resonances can be avoided by satisfying

$$\xi_1 > 1/2, \tag{6.14}$$

as noted above, but this has the unfortunate consequence that physical resonance is also spuriously suppressed.

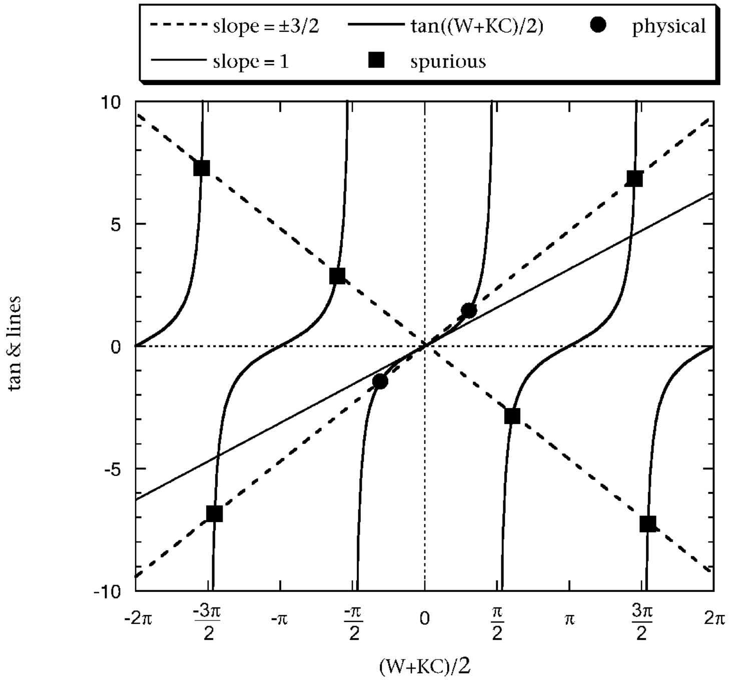

Figure 1. Schematic for solutions of (6.11) when $|\alpha/(\Omega_k + kU)| \geqslant 1$. Resonance occurs at intersections of $y = \tan\{(W + KC)/2\}$ and $y = \pm\{\alpha/(\Omega_k + kU)\}\{(W + KC)/2\}$. The limiting line $y = (W + KC)/2$ is plotted for reference.

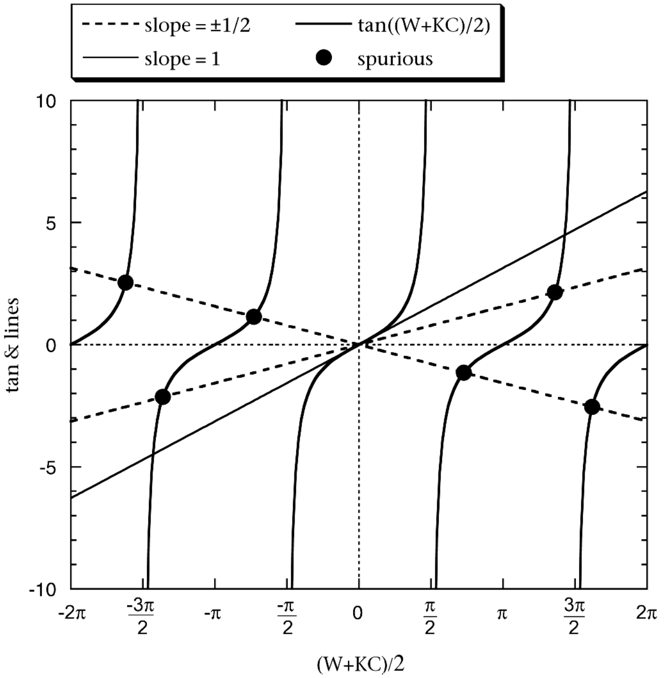

Figure 2. As in Fig. 1 except for $|\alpha/(\Omega_k + kU)| < 1$.

7. EFFECTS OF DAMPING DUE TO INTERPOLATION

The damping due to interpolation can be incorporated into the analysis by introducing the response function ρ of the adopted interpolator (see e.g. Bates and McDonald (1982) and Gravel *et al.* (1993) for further details of interpolation response functions and their use). This is achieved by formally multiplying all the terms evaluated at upstream points $(x - a)$ in the discretized equations of section 3 by ρ, and then carrying them through in the analysis. Exact interpolation corresponds to setting ρ equal to unity but, in general, $|\rho| < 1$ and interpolation gives rise to damping. This damping, e.g. with cubic interpolation, is usually negligible for large and medium scales but becomes progressively stronger at decreasingly small scale. To facilitate insight and to keep the analysis tractable, the damping effect of interpolation has been neglected in all the above analysis by assuming $\rho \equiv 1$, i.e. perfect interpolation. However, some comments are now made regarding the impact that the incorporation of the damping induced by interpolation has on both the above analysis and on the conclusions drawn therefrom.

For the stability and accuracy analysis of section 4, the discrete dispersion relations given in column 2 of Table 1 are all multiplied by ρ. Because $|\rho| \leqslant 1$, this enhances stability since it serves to reduce $|E|$ and to make it easier to achieve stability, particularly at small scales where discrete schemes are often most unstable and where the effect of damping is strongest. The effect on accuracy is to introduce terms of $O(\Delta x^m)$ where m is determined by the order of accuracy of the adopted interpolator.

For the forced non-resonant response, discussed in section 5, (5.2) for the implicit coupling becomes

$$(F_k^{\text{forced}})_{\text{implicit}} = \frac{\{\xi_3 \mathcal{E} + (1 - \xi_3)\rho\} \Delta t R_k}{(1 + i\alpha\Delta t\xi_1 + \beta\Delta t\xi_2)\mathcal{E} - \{1 - i\alpha\Delta t(1 - \xi_1) - \beta\Delta t(1 - \xi_2)\}\rho}. \tag{7.1}$$

The analogous expressions for the other couplings may be obtained from Table 3 by formally dividing $\mathcal{E}$ wherever it appears by the interpolation response function ρ, and by then multiplying both the numerator and denominator by ρ to avoid its possibly singular behaviour (e.g. at the smallest resolvable scale where it might be zero). The effect on accuracy is to introduce terms of $O(\Delta x^m)$, where m is determined by the order of accuracy of the adopted interpolator, into the truncation errors displayed in Table 4 and the exact steady-state solution is then no longer exactly recovered for any of the coupling schemes.

For the forced resonant response analysis of section 6, the right-hand sides of (6.1) and (6.3) are respectively multiplied by ρ and ρ^2. The consequence is that if $|\rho| < 1$, i.e. the interpolation is not exact (the case of exact interpolation, when $\rho = 1$, has already been considered in section 6), then this is sufficient to suppress resonance since the right-hand side of (6.3) can no longer be on the unit circle when the off-centring parameters obey the robust stability constraint (4.6). A similar reasoning applied to the analogous relations for the other couplings leads to the same conclusion.

8. SUMMARY AND CONCLUSION

A framework for examining the numerics of physics–dynamics coupling has been presented. To illustrate the usefulness of this methodology, four couplings have been analysed in the framework of a semi-implicit semi-Lagrangian dynamical core and compared under the assumption, for the most part, that the associated interpolations are performed exactly.

The explicit coupling has the advantages of simplicity (this is particularly attractive in a full model where the highly nonlinear physics terms can be efficiently evaluated in an explicit manner) and correct representation of the exact steady-state solution for constant forcing. However, it suffers from the important disadvantage that the time step is limited by both the stability condition (4.7), which is very restrictive for fast processes such as vertical diffusion of the boundary layer at high resolution, and the $O(\Delta t)$ discretization of the $-\beta F$ damping term.

The implicit coupling addresses the stability issue of the explicit coupling while still correctly representing the exact steady-state solution for constant forcing, but has the disadvantage that the simultaneous implicit coupling of the physics and dynamics leads to a highly nonlinear and computationally difficult and expensive problem to solve.

The split-implicit coupling is a dynamics predictor followed by a physics corrector. It addresses the stability issue of the explicit coupling while keeping the physics discretization distinct from the dynamics discretization but, as argued by CLZ98, it does so at the expense of accuracy—one $O(\Delta t)$ discretization has effectively been replaced by another, and the resulting truncation error is still large for large Δt. In particular, the split-implicit coupling corrupts the steady-state solution and the forced response can, as identified by CLZ98, be spuriously amplified by an order of magnitude. However, a strong time-implicitly treated damping mechanism, such as vertical diffusion, can significantly decrease the seriousness of this latter problem to the point of even underestimating the forced response.

The symmetrized split-implicit coupling is composed of two physics discretizations symmetrically arranged around a dynamics sub-step. It addresses the stability and accuracy deficiencies of the explicit coupling while still correctly representing the exact steady-state solution for constant forcing, and it also has the virtue of keeping the physics discretization distinct from the dynamics one. It partially shares the disadvantage of the implicit coupling inasmuch as the second physics sub-step is an implicit discretization of the highly nonlinear physics. However, the usual column-based physical parametrizations are such that the discrete set of nonlinear equations can be solved column by column, i.e. the nonlinearity appears only in the vertical direction rather than in all three directions simultaneously as it does for the implicit coupling.

Stationary spatial forcing, such as that caused by orography, can lead to spurious computational resonance. The analysis presented here considers the impact of a time-dependent spatial forcing. It is found that this too can lead to spurious computational resonance. Further, in the absence of any controlling mechanism, such as discussed below, this resonance can be avoided only by placing a restriction on the time step which is twice as restrictive as that needed to avoid spurious resonance caused by stationary forcing.

For all the examined couplings, spurious computational resonance can occur when $\xi_1 = 1/2$, i.e. for a centred discretization of the semi-implicitly treated terms, or when $\alpha \Delta t = 0$. In both cases, for time-dependent forcing, i.e. $\Omega_k \neq 0$, resonance can be avoided by setting $0 < \beta \Delta t < 2$, i.e. by applying some diffusion. Decentring the semi-implicitly treated terms, i.e. by setting $\xi_1 > 1/2$, removes the possibility of resonance when $\alpha \Delta t \neq 0$. However, this has the unfortunate consequence of spuriously inhibiting physical resonance, when it exists. For the special case of stationary forcing, respect of the Courant condition (6.13) suppresses spurious resonance when $\xi_1 = 1/2$ while still capturing physical resonance.

The incorporation of the damping due to the interpolation associated with semi-implicit semi-Lagrangian discretizations was also considered. The impact of interpolation is for the most part minor: it introduces a spatial truncation error, whose order is determined by the order of the interpolation error, and this introduces a damping, which is negligible at large and medium spatial scales but which can be important at small spatial scale; computational stability is enhanced, particularly at small spatial scale. Arguably, the most important impact of interpolation is that, in addition to the real resonance, it inhibits spurious computational resonance, and reduces the need for explicit damping and off-centring, two alternative ways of controlling this.

The presented framework for examining the numerics of physics–dynamics coupling is fairly general and could be adapted for other applications. A current thrust of 4-D variational data assimilation (Janisková *et al.* 1999b) is to include a simplified linear physics package (Janisková *et al.* 1999a) into the tangent linear and adjoint models used in the incremental variational formulation. The numerical analysis framework presented here could be adapted to examine relevant issues for this, such as the possibility that specific physics–dynamics couplings could lead to the introduction of a systematic bias into the data assimilation procedure. Gordon *et al.* (2000) showed that the poor circulation in the ocean component of an early version of the Hadley Centre's coupled climate-model led to a spurious climate-drift, and that this had to be controlled by a flux adjustment procedure to obtain a realistic simulated climate. It is possible that the numerics of atmosphere–ocean couplings might also give rise to a small but nevertheless non-negligible climate-drift. The present analysis could perhaps be adapted to provide some insight into whether this is likely or not. Finally, a further possible application would be to examine the impact of incorporating the physics as a sequence of predictor–corrector steps of physical sub-processes (turbulent diffusion, surface exchanges, clouds, radiation, convection etc.), and the order of doing so, versus a simultaneous evaluation of all process tendencies using the same atmospheric state.

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