Q. J. R. Meteorol. Soc. (2006), **132**, pp. 27–42 doi: 10.1256/qj.05.49

Some numerical properties of approaches to physics–dynamics coupling for NWP

By MARK DUBAL*, NIGEL WOOD and ANDREW STANIFORTH
Met Office, Exeter, UK

(Received 23 March 2005; revised 11 July 2005)

SUMMARY

At the present time there exist a number of different approaches to the problem of coupling parametrized physical processes to the dynamical core in operational numerical weather-prediction (NWP) and climate models. Motivated by the various strategies in use, some idealized representative coupling schemes are constructed and subsequently analysed using a methodology in which the physics and dynamics terms are represented in a simplified way. Particular numerical properties of the idealized schemes which are of interest are the ability to capture correct steady-state solutions and to be second-order accurate in time. In general, the schemes require specific choices for the time-differencing of certain coupled processes if correct steady-state solutions are to be obtained. This has implications for the overall numerical stability of a coupling strategy. An alternative physics–dynamics coupling approach is then described and analysed. A multiple-sweep predictor–corrector coupling scheme is shown to capture the correct steady-state solution and to allow for second-order accuracy, provided that the convective process is coupled explicitly. This approach has a number of advantages over those currently used in operational NWP models.

KEYWORDS: Numerical analysis Parallel-split coupling Physics parametrizations Predictor–corrector Sequential-split coupling

1. INTRODUCTION

The coupling of physical parametrization processes to the underlying dynamical core is an important component of all numerical weather-prediction (NWP) and climate models. In recent years, increasing attention has been paid to the numerics of the coupling procedure, since forecast results have been shown to be sensitive to the particular strategy employed (see e.g. Caya *et al.* 1998; Williamson 2002; Cullen and Salmond 2003; Martínez 2004; Beljaars *et al.* 2004). Staniforth *et al.* (2002a,b) (henceforth denoted by SWC) developed a simple methodology for analysing the numerics of different physics–dynamics coupling approaches, and they subsequently applied it to determine the properties of a number of time-split coupling schemes. Further applications of the SWC methodology have been made by Dubal *et al.* (2004) to investigate coupling schemes when the physics processes have very different timescales, and by Dubal *et al.* (2005) for more complex cases where multiple physics processes and multiple timescales are present. In these two papers the numerical consequences of utilizing certain general approaches, such as parallel-split or sequential-split coupling of the parametrized processes, were compared and contrasted. Explicit differencing with parallel-splitting (equivalent to no splitting) was found to be useful for coupling slow processes. However, fast processes do require some degree of implicit differencing combined with a sequential-splitting approach for numerically stable and accurate coupling schemes. Then, the stability and accuracy is dependent on the order in which the fast processes are coupled in the sequential-split operation. Generally the fastest process ought to be coupled last. These works indicate that the methodology developed by SWC can provide valuable insight into the numerical properties of physics–dynamics coupling approaches for full NWP models. Here, specific coupling models, considered to be paradigms of actual approaches used in NWP models, are investigated. The purpose of the present work is two-fold.

* Corresponding author: Met Office, FitzRoy Road, Exeter EX1 3PB UK. e-mail: mark.dubal@metoffice.gov.uk

First, the SWC methodology is used to elucidate the numerical properties of various idealized coupling schemes, the forms of which are motivated by the approaches adopted in current operational NWP models. A particular emphasis of the investigation is on the ability of these schemes to capture correctly steady-state solutions, regardless of the values of any time-off-centring weights in the difference equations and, additionally, the possibility for the schemes to be second-order accurate in time. Despite being somewhat ideal in their nature, these two properties are important. A correct steady-state implies a correct balance between individual physics and dynamics processes, thereby reducing systematic errors and biases. If the need to capture a correct steady-state solution imposes restrictions on the time-differencing used, then this can have implications for the numerical stability of the coupling scheme. With regard to accuracy, a first-order-accurate coupling scheme implies a first-order-accurate solution, even if both the dynamics and physics packages are accurate to second order.

Second, the SWC methodology is used to analyse a multiple-sweep predictor–corrector coupling scheme which can address both the steady-state and second-order accuracy issues, subject to some restrictions.

In the following section the NWP equations are written, following the SWC methodology, as a canonical model problem. The model problem includes linear dynamics, a physics forcing term and two linear dissipative physics terms. One of the linear physics terms is taken to represent fast boundary-layer type processes, while the other represents convective-like processes. In section 3, various idealized coupling schemes, which are representative of particular approaches to physics–dynamics coupling in operational NWP and climate models, are constructed and used to solve the canonical problem of section 2. The schemes idealize coupling approaches, which can be categorized as: (a) purely sequential-split; (b) mostly parallel-split; (c) mixed sequential/parallel-split; and (d) sequential-split with predictors. In sections 4 and 5, respectively, the steady-state and accuracy properties of the idealized coupling schemes are investigated. Section 6 describes and analyses an improved coupling scheme. This scheme attempts to meet the two requirements of obtaining a correct steady-state solution independently of the values of time-off-centring weights, and of allowing for a second-order-accurate discretization of the coupled physics–dynamics system. Some conclusions are presented in section 7.

2. Model equation

To facilitate a tractable analysis of physics–dynamics coupling schemes, the SWC methodology represents the full NWP equations in terms of a canonical model problem. The solutions to this problem can be regarded as representing the normal modes of the more complete system (see Eqs. (2.1) and (2.3) of SWC). For an analysis of a typical NWP model, a suitable model equation in SWC form is,

$$\frac{\mathrm{D}F(x,t)}{\mathrm{D}t} = -\mathrm{i}\alpha F(x,t) - \beta F(x,t) - \gamma F(x,t) + R_k\, \mathrm{e}^{\mathrm{i}(kx+\Omega_k t)}, \tag{1}$$

where,

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + U\frac{\partial}{\partial x}. \tag{2}$$

Quantities α, $\beta > 0$ and $\gamma > 0$ in (1) are real and constant. In addition the advective velocity $U \geq 0$ is taken to be constant. The oscillatory forcing term, $R_k \exp\{\mathrm{i}(kx+\Omega_k t)\}$, has a wave number, k, and frequency, Ω_k, and is taken to be independent of $F(x,t)$. The forcing amplitude, R_k, can be a complex number. The solution, $F(x,t)$, of (1) represents

the kth normal mode of the full NWP equation set, and will have a spatial dependency proportional to $\exp(\mathrm{i}kx)$ (see the discussion by SWC).

In (1), the $-\mathrm{i}\alpha F$ term models oscillatory dynamical processes such as those that govern acoustic and gravity-wave propagation. The first dissipative term, $-\beta F$, is taken to represent vertical diffusion (boundary-layer) effects, which can be fast. The second dissipative term, $-\gamma F$, is associated with convective-like processes. The time-dependent forcing, $R_k \exp\{\mathrm{i}(kx + \Omega_k t)\}$, can represent a slow forcing (such as the diurnal cycle of radiation) or a fast one (such as variation in radiation due to changes in cloud cover). In this work the forcing is generally taken to be representative of the slow physics processes.

When $\Omega_k = 0$, i.e. a time-independent forcing, and in the limit $t \to \infty$, (1) has the steady-state solution,

$$F_{\mathrm{ss}} = \frac{R_k \, \mathrm{e}^{\mathrm{i}kx}}{\beta + \gamma + \mathrm{i}(\alpha + kU)}. \tag{3}$$

The exact free solution to (1) can be found by setting $R_k = 0$ and seeking solutions of the form $F = f(t)\exp(\mathrm{i}kx)$. Then,

$$F^{\mathrm{free}}_{\mathrm{exact}} = F^{\mathrm{free}}_k \, \mathrm{e}^{\mathrm{i}k(x-Ut)} \, \mathrm{e}^{-\mathrm{i}\alpha t} \, \mathrm{e}^{-\beta t} \, \mathrm{e}^{-\gamma t}, \tag{4}$$

where F^{free}_k is the amplitude at $t = 0$ of the kth component of the free solution.

From (4), the free response function for (1) can be written as,

$$E_{\mathrm{exact}} \equiv \frac{F^{\mathrm{free}}_{\mathrm{exact}}(t + \Delta t)}{F^{\mathrm{free}}_{\mathrm{exact}}(t)} = \mathrm{e}^{-\mathrm{i}kU\Delta t} \, \mathrm{e}^{-\mathrm{i}\alpha\Delta t} \, \mathrm{e}^{-\beta\Delta t} \, \mathrm{e}^{-\gamma\Delta t}, \tag{5}$$

where Δt is a time-step interval. For later use, expanding E_{exact} in powers of Δt results in,

$$\begin{aligned} E_{\mathrm{exact}} = {} & 1 - (\mathrm{i}\alpha + \mathrm{i}kU + \beta + \gamma)\Delta t \\ & + \left\{ \frac{(\beta + \gamma)^2 - \alpha^2}{2} + \mathrm{i}\alpha(\beta + \gamma) + \mathrm{i}kU(\beta + \gamma + \mathrm{i}\alpha) - \frac{(kU)^2}{2} \right\} \Delta t^2 \\ & + \mathrm{O}(\Delta t^3). \end{aligned} \tag{6}$$

3. Coupling schemes for operational models

This section describes idealized coupling schemes that can be thought of as representing the general coupling strategies used in different operational NWP models. The coupling strategies of full NWP models are necessarily complex and often employ subtle techniques, particularly for nonlinear terms that cannot be analysed using the linear SWC methodology. Despite this, a linear analysis can be useful in identifying approaches that are unlikely to perform well in a nonlinear regime, i.e. it can be used as a starting point to establish necessary, but not sufficient, properties of a good coupling scheme.

The aim of the analysis shown below is to determine to what degree the various coupling schemes can meet the two requirements of obtaining a correct steady-state solution, independently of any time-off-centring weights, and of allowing for a coupling that is second-order accurate in time. The non-advecting steady-state solution, (3) with

$U = 0$, and the expanded free response function (6) for the model equation (1), provide reference solutions.

In the following descriptions of the numerical coupling schemes, the superscript n labels the time level, subscript d denotes evaluation at the departure point, Δt is the time step and the ξ_j's are time-off-centring weights, the values of which normally determine the 'implicitness' of the coupling (any exceptions are noted). Quantities without the d subscript are evaluated at grid points. Note that $\overline{\xi}_j \equiv 1 - \xi_j$, and usually $0 \leq \xi_j \leq 1$. Intermediate temporal values are denoted by $F^{(1)}$ etc. Moreover, for the purposes of generality, the slow physics term, $R \equiv R_k \exp\{\mathrm{i}(kx + \Omega_k t)\}$, will also often be assigned intermediate temporal values (e.g. $R^{(1)}$ etc.) as if it were evaluated using an intermediate value of F (e.g. $F^{(1)}$), despite it here being assumed to be independent of F.

Each process in (1) has one or more associated time-off-centring weights. For the schemes shown below, weight ξ_1 is associated with the dynamical term $-\mathrm{i}\alpha F$. Weights ξ_2 and ξ_5 are associated with the fast boundary-layer process, $-\beta F$. Weights ξ_3 and ξ_6 are associated with the slow physics term, R. Finally, weights ξ_4 and ξ_7 are associated with the convective-like process, $-\gamma F$.

(*a*) *Scheme A: Purely sequential-split approach*

A popular approach is the purely sequential-split method, see Caya *et al.* (1998), for example, for a discussion. This motivates the coupling scheme,

$$\frac{F^{(1)} - F^n_{\mathrm{d}}}{\Delta t} = -\mathrm{i}\alpha(\xi_1 F^{(1)} + \overline{\xi}_1 F^n_{\mathrm{d}}), \tag{7}$$

$$\frac{F^{(2)} - F^{(1)}}{\Delta t} = R^n, \tag{8}$$

$$\frac{F^{(3)} - F^{(2)}}{\Delta t} = -\beta(\xi_2 F^{(3)} + \overline{\xi}_2 F^n), \tag{9}$$

$$\frac{F^{n+1} - F^{(3)}}{\Delta t} = -\gamma(\xi_4 F^{n+1} + \overline{\xi}_4 F^n), \tag{10}$$

for solving (1). This is a sequential-split implementation of coupling with the ordering: dynamics, slow physics, boundary-layer processes and convection. A semi-implicit semi-Lagrangian (SISL) formulation is used to evaluate the dynamical tendency (7). The physics tendencies are evaluated using time-averages with grid-point values, F^n. It is possible to use intermediate temporal values of F in the evaluation of the physics tendencies, e.g. use $-\beta(\xi_2 F^{(3)} + \overline{\xi}_2 F^{(2)})$ rather than $-\beta(\xi_2 F^{(3)} + \overline{\xi}_2 F^n)$ on the right-hand side of (9). Doing so generally leads to an increase in splitting error, see e.g. Beljaars *et al.* (2004) and Dubal *et al.* (2005), and therefore this option is not considered here. Alternatively, using departure point values, F^n_{d}, rather than F^n, i.e. an averaging along the semi-Lagrangian trajectory, so that, for example, the right-hand side of (9) becomes $-\beta(\xi_2 F^{(3)} + \overline{\xi}_2 F^n_{\mathrm{d}})$, can lead to improvements in accuracy (see the discussion in section 5 below). However, operational models using the purely sequential-split strategy tend not to adopt this averaging technique.

The slow physics term, R, is treated in an explicit first-order-accurate fashion. For the purpose of generality, the convective-like process, $-\gamma F$, is initially treated in a time-off-centred implicit fashion. This can later be specialized to explicit differencing, as required. Generally, the fast boundary-layer process, $-\beta F$, will need to be treated in an implicit fashion ($\xi_2 > 0$) to ensure numerical stability.

By eliminating all intermediate values of F, it can be shown that the system (7)–(10) is equivalent to the single coupling equation,

$$\frac{F^{n+1}-F_{\mathrm{d}}^{n}}{\Delta t} = -\mathrm{i}\alpha(\xi_1 F^{n+1} + \overline{\xi}_1 F_{\mathrm{d}}^{n}) - \beta(\xi_2 F^{n+1} + \overline{\xi}_2 F^{n}) - \gamma(\xi_4 F^{n+1} + \overline{\xi}_4 F^{n}) + R^{n} - \chi_{\mathrm{A}_1} - \chi_{\mathrm{A}_2}, \tag{11}$$

where

$$\chi_{\mathrm{A}_1} \equiv \{\mathrm{i}\alpha\beta\xi_1(\xi_2 F^{n+1} + \overline{\xi}_2 F^{n}) + \gamma(\mathrm{i}\alpha\xi_1 + \beta\xi_2)(\xi_4 F^{n+1} + \overline{\xi}_4 F^{n}) - \mathrm{i}\alpha\xi_1 R^{n}\}\Delta t, \tag{12}$$

and

$$\chi_{\mathrm{A}_2} \equiv \mathrm{i}\alpha\beta\gamma\xi_1\xi_2(\xi_4 F^{n+1} + \overline{\xi}_4 F^{n})\Delta t^2, \tag{13}$$

are splitting error terms.

(*b*) *Scheme B: Mostly parallel-split approach*

One form of the NCAR* Community Climate Model-3 (CCM3) coupling approach, as described by Williamson (2002), motivates the coupling scheme,

$$\frac{F^{(1)}-F_{\mathrm{d}}^{n}}{\Delta t} = -\mathrm{i}\alpha(\xi_1 F^{(1)} + \overline{\xi}_1 F_{\mathrm{d}}^{n}), \tag{14}$$

$$\frac{F^{(2)}-F^{n}}{\Delta t} = -\beta(\xi_2 F^{(2)} + \overline{\xi}_2 F^{n}), \tag{15}$$

$$\frac{F^{(3)}-F^{n}}{\Delta t} = \left[\frac{F^{(1)}-F_{\mathrm{d}}^{n}}{\Delta t}\right]_{\alpha} + \left[\frac{F^{(2)}-F^{n}}{\Delta t}\right]_{\beta} + R^{n}, \tag{16}$$

$$\frac{F^{n+1}-F^{(3)}}{\Delta t} = -\gamma(\xi_4 F^{n+1} + \overline{\xi}_4 F^{n}). \tag{17}$$

In the above,

$$\left[\frac{F^{(1)}-F_{\mathrm{d}}^{n}}{\Delta t}\right]_{\alpha} = -\frac{\mathrm{i}\alpha F_{\mathrm{d}}^{n}}{1+\mathrm{i}\alpha\Delta t\xi_1}, \tag{18}$$

from (14), and

$$\left[\frac{F^{(2)}-F^{n}}{\Delta t}\right]_{\beta} = -\frac{\beta F^{n}}{1+\beta\Delta t\xi_2}, \tag{19}$$

from (15). Again, the dynamical tendency uses a SISL formulation, while the physics tendencies use grid-point values, F^n, for the same reasons as noted previously.

This scheme employs a parallel-split coupling of the dynamics, the boundary-layer processes and the slow physics. The convective-like process (represented by $-\gamma F$) is coupled sequentially after the parallel-split procedure.

If intermediate values of F are eliminated from (14)–(17), then the equivalent single coupling equation is,

$$\frac{F^{n+1}-F^{n}}{\Delta t} = -\mathrm{i}\alpha\{\xi_1(F^{n+1}-F^{n}) + F_{\mathrm{d}}^{n}\} - \beta(\xi_2 F^{n+1} + \overline{\xi}_2 F^{n}) - \gamma(\xi_4 F^{n+1} + \overline{\xi}_4 F^{n}) + R^{n} - \chi_{\mathrm{B}_1} - \chi_{\mathrm{B}_2}, \tag{20}$$

* National Center for Atmospheric Research.

where

$$\chi_{\mathrm{B}_1} \equiv [\mathrm{i}\alpha\beta\xi_1\xi_2(F^{n+1} - F^n) + \mathrm{i}\alpha\beta(\xi_1 F^n + \xi_2 F_{\mathrm{d}}^n) + (\mathrm{i}\alpha\xi_1 + \beta\xi_2)\{\gamma(\xi_4 F^{n+1} + \overline{\xi}_4 F^n) - R^n\}]\Delta t, \tag{21}$$

and

$$\chi_{\mathrm{B}_2} \equiv \mathrm{i}\alpha\beta\xi_1\xi_2\{\gamma(\xi_4 F^{n+1} + \overline{\xi}_4 F^n) - R^n\}\Delta t^2, \tag{22}$$

are splitting error terms.

(*c*) *Scheme C: Mixed sequential/parallel-split approach*

The current Met Office Unified Model (UM) coupling scheme for solving (1) motivates the following SWC equations,

$$\frac{F^{(1)} - F_{\mathrm{d}}^n}{\Delta t} = -\mathrm{i}\alpha(\xi_1 F^n + \overline{\xi}_1 F_{\mathrm{d}}^n) + R_{\mathrm{d}}^n, \tag{23}$$

$$\frac{F^{(2)} - F^{(1)}}{\Delta t} = -\gamma(\xi_4 F^{(2)} + \overline{\xi}_4 F^n), \tag{24}$$

$$\frac{F^{(3)} - F^{(2)}}{\Delta t} = -\beta(\xi_2 F^{(3)} + \overline{\xi}_2 F^n), \tag{25}$$

$$\frac{F^{n+1} - F^{(3)}}{\Delta t} = -\mathrm{i}\alpha\xi_1(F^{n+1} - F^n). \tag{26}$$

The first equation, (23), is an explicit-dynamics predictor, and the final one, (26), is an implicit dynamics corrector, which is representative of the Helmholtz type equation that typically appears in full NWP models. For reasons of generality, (24) indicates an implicit coupling of the convection—but, in fact, the current UM scheme treats $-\gamma F$ mostly in an explicit way, i.e. it uses $\xi_4 = 0$. The fast boundary-layer process, $-\beta F$, is treated in an implicit manner for reasons of numerical stability. Note that the order of coupling for the convection and boundary-layer processes is the reverse of that used in the previous schemes A and B. As discussed in the next section, this ordering has consequences for a scheme's ability to capture correctly the steady-state solution whilst retaining numerical stability.

If all intermediate values of F are eliminated from (23)–(26) then the system is equivalent to the single coupling equation,

$$\frac{F^{n+1} - F_{\mathrm{d}}^n}{\Delta t} = -\mathrm{i}\alpha(\xi_1 F^{n+1} + \overline{\xi}_1 F_{\mathrm{d}}^n) - \beta(\xi_2 F^{n+1} + \overline{\xi}_2 F^n) - \gamma(\xi_4 F^{n+1} + \overline{\xi}_4 F^n) + R_{\mathrm{d}}^n - \chi_{\mathrm{C}_1} - \chi_{\mathrm{C}_2}, \tag{27}$$

where,

$$\chi_{\mathrm{C}_1} \equiv \{\mathrm{i}\alpha\xi_1(\beta\xi_2 + \gamma\xi_4)(F^{n+1} - F^n) + \beta\gamma\xi_4(\xi_2 F^{n+1} + \overline{\xi}_2 F^n)\}\Delta t, \tag{28}$$

and

$$\chi_{\mathrm{C}_2} \equiv \mathrm{i}\alpha\beta\gamma\xi_1\xi_2\xi_4(F^{n+1} - F^n)\Delta t^2, \tag{29}$$

are splitting error terms.

(*d*) *Scheme D: Sequential-split with predictor approach*

The semi-Lagrangian averaging of physical parametrizations (SLAVEPP), as described and implemented by Wedi (1999) in the European Centre for Medium-Range Weather Forecasts (ECMWF) model, uses a predictor approach when coupling the physics processes. This is to try to reduce dependencies on the order in which the physics processes are included, and on the size of time step. The SLAVEPP coupling approach motivates the following scheme:

$$\frac{F^{(1)} - F_{\mathrm{d}}^{n}}{\Delta t} = -\mathrm{i}\alpha(\xi_1 F^{n} + \overline{\xi}_1 F_{\mathrm{d}}^{n}), \tag{30}$$

$$\frac{F^{(2)} - F^{(1)}}{\Delta t} = -\beta(\xi_2 F^{(2)} + \overline{\xi}_2 F_{\mathrm{d}}^{n}) + \xi_3 R^{n} + \overline{\xi}_3 R_{\mathrm{d}}^{n}, \tag{31}$$

$$\frac{F^{(3)} - F^{(2)}}{\Delta t} = -\gamma(\xi_4 F^{(P)} + \overline{\xi}_4 F_{\mathrm{d}}^{(P)n-1}), \tag{32}$$

$$\frac{F^{n+1} - F^{(3)}}{\Delta t} = -\mathrm{i}\alpha\xi_1(F^{n+1} - F^{n}), \tag{33}$$

where

$$\begin{aligned}\frac{F^{(P)} - F^{(1)}}{\Delta t} = &-\beta(\xi_5 F^{(P)} + \overline{\xi}_5 F_{\mathrm{d}}^{n}) - \gamma(\xi_7 F^{n} + \overline{\xi}_7 F_{\mathrm{d}}^{n-1}) \\ &+ \xi_6 R^{n} + \overline{\xi}_6 R_{\mathrm{d}}^{n-1}.\end{aligned} \tag{34}$$

Equation (34) is a predictor designed to reduce the ordering and time-step dependency of the sequential splitting. In general, the physics processes are coupled by averaging along the semi-Lagrangian trajectory (see the SLAVEPP method of Wedi 1999). Note that $F_{\mathrm{d}}^{(P)n-1}$ is the predictor value from the previous time level, $n-1$, which has been interpolated to the departure point. Using (34) it is defined as,

$$\begin{aligned}\frac{F_{\mathrm{d}}^{(P)n-1} - F_{\mathrm{d}}^{(1)n-1}}{\Delta t} &= \{-\beta(\xi_5 F^{(P)n-1} + \overline{\xi}_5 F_{\mathrm{d}}^{n-1}) - \gamma(\xi_7 F^{n-1} + \overline{\xi}_7 F_{\mathrm{d}}^{n-2}) \\ &\quad + \xi_6 R^{n-1} + \overline{\xi}_6 R_{\mathrm{d}}^{n-2}\}_{\mathrm{d}} \\ &= -\beta(\xi_5 F_{\mathrm{d}}^{(P)n-1} + \overline{\xi}_5 F_{\mathrm{dd}}^{n-1}) - \gamma(\xi_7 F_{\mathrm{d}}^{n-1} + \overline{\xi}_7 F_{\mathrm{dd}}^{n-2}) \\ &\quad + \xi_6 R_{\mathrm{d}}^{n-1} + \overline{\xi}_6 R_{\mathrm{dd}}^{n-2},\end{aligned} \tag{35}$$

where, using (30),

$$\begin{aligned}\frac{F_{\mathrm{d}}^{(1)n-1} - F_{\mathrm{dd}}^{n-1}}{\Delta t} &= \{-\mathrm{i}\alpha(\xi_1 F^{n-1} + \overline{\xi}_1 F_{\mathrm{d}}^{n-1})\}_{\mathrm{d}} \\ &= -\mathrm{i}\alpha(\xi_1 F_{\mathrm{d}}^{n-1} + \overline{\xi}_1 F_{\mathrm{dd}}^{n-1}).\end{aligned} \tag{36}$$

The dd subscript denotes a quantity at a grid point which has been evaluated at a departure point, and which is subsequently further interpolated to a departure point at a new time level.

By using predictor values, the convection term, $-\gamma F$, is never coupled implicitly regardless of the values of its associated weights ξ_4 and ξ_7. As for scheme C, a dynamics predictor, (30), is used first and the final equation is an implicit dynamics corrector, (33), which represents a Helmholtz-type equation.

If all intermediate values of F are eliminated from (30)–(36) the result is,

$$\frac{F^{n+1}-F_{\rm d}^{n}}{\Delta t} = -{\rm i}\alpha(\xi_1 F^{n+1}+\overline{\xi}_1 F_{\rm d}^{n}) - \beta\{(\xi_2+\xi_5)F^{n+1}+(\overline{\xi}_2-\xi_5)F_{\rm d}^{n}\} - \gamma(\xi_4 F_{\rm d}^{n}+\overline{\xi}_4 F_{\rm dd}^{n-1})+\xi_3 R^{n}+\overline{\xi}_3 R_{\rm d}^{n}-\chi_{\rm D_1}-\chi_{\rm D_2}, \tag{37}$$

where,

$$\begin{aligned}\chi_{\rm D_1} \equiv\ & [{\rm i}\alpha\beta\xi_1\xi_2(F^{n+1}-F^{n})+\beta\gamma(\xi_2-\overline{\xi}_5)(\xi_4 F_{\rm d}^{n}+\overline{\xi}_4 F_{\rm dd}^{n-1})\\ & +\beta\xi_5\{{\rm i}\alpha(\xi_1 F^{n+1}+\overline{\xi}_1 F_{\rm d}^{n})+\beta(\xi_2 F^{n+1}+\overline{\xi}_2 F_{\rm d}^{n})-\xi_3 R^{n}-\overline{\xi}_3 R_{\rm d}^{n}\}\\ & -{\rm i}\alpha\gamma\{\xi_1(\xi_4 F^{n}+\overline{\xi}_4 F_{\rm d}^{n-1})+\overline{\xi}_1(\xi_4 F_{\rm d}^{n}+\overline{\xi}_4 F_{\rm dd}^{n-1})\}\\ & -\gamma^2\{\xi_4\xi_7 F^{n}+(\xi_4\overline{\xi}_7+\overline{\xi}_4\xi_7)F_{\rm d}^{n-1}+\overline{\xi}_4\overline{\xi}_7 F_{\rm dd}^{n-2}\}\\ & +\gamma\{\xi_4\xi_6 R^{n}+(\xi_4\overline{\xi}_6+\overline{\xi}_4\xi_6)R_{\rm d}^{n-1}+\overline{\xi}_4\overline{\xi}_6 R_{\rm dd}^{n-2}\}]\Delta t,\end{aligned} \tag{38}$$

and

$$\begin{aligned}\chi_{\rm D_2} \equiv\ & \beta\xi_2[{\rm i}\alpha\beta\xi_1\xi_5(F^{n+1}-F^{n})-\beta\gamma\overline{\xi}_5(\xi_4 F_{\rm d}^{n}+\overline{\xi}_4 F_{\rm dd}^{n-1})\\ & -{\rm i}\alpha\gamma\{\xi_1(\xi_4 F^{n}+\overline{\xi}_4 F_{\rm d}^{n-1})+\overline{\xi}_1(\xi_4 F_{\rm d}^{n}+\overline{\xi}_4 F_{\rm dd}^{n-1})\}\\ & -\gamma^2\{\xi_4\xi_7 F^{n}+(\xi_4\overline{\xi}_7+\overline{\xi}_4\xi_7)F_{\rm d}^{n-1}+\overline{\xi}_4\overline{\xi}_7 F_{\rm dd}^{n-2}\}\\ & +\gamma\{\xi_4\xi_6 R^{n}+(\xi_4\overline{\xi}_6+\overline{\xi}_4\xi_6)R_{\rm d}^{n-1}+\overline{\xi}_4\overline{\xi}_6 R_{\rm dd}^{n-2}\}]\Delta t^2,\end{aligned} \tag{39}$$

are splitting error terms.

4. Steady-state solutions

In the absence of advection ($U=0$), the steady-state behaviour of the schemes described above can be determined by: (i) setting F at every grid point or departure point, on all time levels, to a steady-state value, $F'_{\rm ss}$, for each of the single coupling equations listed in section 3 above; and (ii) similarly, setting R at every grid point or departure point, on all time levels, to the value $R_k\,{\rm e}^{{\rm i}kx}$ for each of the single coupling equations. As an example, the steady-state behaviour of coupling scheme C (mixed sequential/parallel splitting) can be determined by setting $F^{n+1}=F_{\rm d}^{n}=F^{n}=F'_{\rm ss}$ and $R_{\rm d}^{n}=R_k\,{\rm e}^{{\rm i}kx}$ in (27)–(29). If the resultant equation is satisfied when $F'_{\rm ss}=F_{\rm ss}$, then the coupling scheme can capture the true (non-advecting) steady-state solution, (3), with $U=0$. In general, a particular combination of values for the time-off-centring weights will be required if the correct steady-state solution is to be obtained.

Table 1 lists the splitting errors occurring in the idealized NWP coupling schemes of section 3, for a non-advecting steady state. Also listed are values of the off-centring weights required for the schemes to produce the correct steady-state solution, (3) with $U=0$. Values of weights not specified can be arbitrary. A coupling scheme's ability to capture the correct steady-state solution should preferably be independent of the values of any off-centring weights. If this is not the case, then there can be a conflict between the stability of the coupling scheme and the accuracy of the steady-state solution obtained (see the discussion below). For all the schemes A–D, any errors in the steady-state solution will be a function of the time step and the timescales of the various dynamics and physics processes involved.

TABLE 1. SPLITTING ERRORS OCCURRING IN THE CASE OF A NON-ADVECTIVE STEADY-STATE ($U = 0$) FOR THE IDEALIZED PHYSICS–DYNAMICS COUPLING SCHEMES OF SECTION 3

Scheme	Splitting errors for the non-advective steady state	Weights for the correct steady state
A	$\chi_{A_1} = \{i\alpha\xi_1(\beta F'_{ss} - R_k\, e^{ikx}) + \gamma(i\alpha\xi_1 + \beta\xi_2)F'_{ss}\}\Delta t$ $\chi_{A_2} = i\alpha\beta\gamma\xi_1\xi_2 F'_{ss}\Delta t^2$	$\xi_1 = \xi_2 = 0$
B	$\chi_{B_1} = \{i\alpha\beta(\xi_1 + \xi_2)F'_{ss} + (i\alpha\xi_1 + \beta\xi_2)(\gamma F'_{ss} - R_k\, e^{ikx})\}\Delta t$ $\chi_{B_2} = i\alpha\beta\xi_1\xi_2(\gamma F'_{ss} - R_k\, e^{ikx})\Delta t^2$	$\xi_1 = \xi_2 = 0$
C	$\chi_{C_1} = \beta\gamma\xi_4 F'_{ss}\Delta t$ $\chi_{C_2} = 0$	$\chi_{C_2} = 0$
D	$\chi_{D_1} = [(\beta\xi_5 - \gamma)\{(i\alpha + \beta + \gamma)F'_{ss} - R_k\, e^{ikx}\} + \beta\gamma\xi_2 F'_{ss}]\Delta t$ $\chi_{D_2} = -\beta\gamma\xi_2\{(i\alpha + \beta\overline{\xi}_5 + \gamma)F'_{ss} - R_k\, e^{ikx}\}\Delta t^2$	$\xi_2 = 0$

Values of the time-off-centring weights required if the scheme is to capture the correct non-advecting steady-state solution, given by (3) with $U = 0$, are listed in the third column. For scheme D the convective term, $-\gamma F$, is always coupled explicitly for any values of ξ_4 or ξ_7.

Table 1 shows that the purely sequential-split coupling scheme, A, produces the correct steady-state solution, and the splitting errors vanish for a steady-state, if $\xi_1 = \xi_2 = 0$. This means treating both the dynamics, $-i\alpha F$, and the fast-physics process, $-\beta F$, with a first-order explicit differencing. This is not viable for stability reasons. Likewise, Table 1 shows that the splitting errors for the mostly parallel-split coupling scheme, B, will vanish, and a correct steady-state solution obtained, if $\xi_1 = \xi_2 = 0$. Thus, for stable versions of coupling schemes A and B ($\xi_2 > 0$), there will be some degree of error in the non-advecting steady-state solution.

For the mixed coupling scheme, C, Table 1 indicates that the requirement for an error-free steady state is simply that $\xi_4 = 0$, i.e. that the convection term, $-\gamma F$, should be coupled using explicit differencing. The current version of the Met Office UM (which motivates this scheme) does, in fact, couple convection in a mostly explicit way. Both the dynamics, $-i\alpha F$, and the fast boundary-layer process, $-\beta F$, can be treated in an implicit manner, which is required for numerical stability. If the convection were coupled after the boundary-layer process (as is done for schemes A and B), then a correct steady-state would require $\xi_2 = 0$, i.e. an explicit treatment of the boundary-layer process, leading to stability problems. Assuming the boundary-layer process is faster than the convection, this reinforces the point that, in a coupling scheme using sequential splitting, the fastest process ought to be coupled last (see Sportisse 2000; Dubal *et al.* 2005).

The situation for coupling scheme D is more complicated (see Table 1). The correct steady-state solution requires that $\xi_2 = 0$. The value of ξ_5 can be arbitrary and, by inspection of (37), can therefore be chosen to obtain a stable and implicit treatment of the fast boundary-layer process, $-\beta F$. Note that, unlike schemes A–C, the convective term, $-\gamma F$, is coupled explicitly for any value of ξ_4. In (37) the convective term, $-\gamma F$, is centred one time level back from the current time level. In fact, four time levels are used in the predictor scheme D. This can introduce spurious computational modes into the solution, as discussed in the following section.

In summary, schemes A and B cannot capture the correct steady-state solution if they are to be stable, that is if the dynamics and the fast boundary-layer process are coupled implicitly. Schemes C and D can be stable and capture the correct steady-state solution, but only if the convective term, $-\gamma F$, is coupled explicitly (this is always the

case for scheme D). Scheme D requires $\xi_2 = 0$, but the time-off-centring weight ξ_5 in the predictor, (34), can be used to couple $-\beta F$ in a stable and accurate fashion. Spurious computational modes may occur for scheme D, as discussed below.

5. NUMERICAL ACCURACY OF THE COUPLING SCHEMES

The numerical accuracy of the idealized coupling schemes can be estimated by comparing their free response functions (expanded in powers of Δt) with the expanded free response function of the model equation, given by (6). To determine the free response functions: (i) set $R = 0$ at all time levels; and (ii) write the departure point values as* e.g. $F_{\mathrm{d}}^n = F^n \, \mathrm{e}^{-\mathrm{i}kU\Delta t}$, in each of the single coupling equations of section 3. Then, seeking solutions of the form $F^n = f_{\mathrm{X}}^n(t) \, \mathrm{e}^{\mathrm{i}kx}$, where X denotes one of the coupling schemes A–D, an expression for the response function is,

$$E_{\mathrm{X}} \equiv \frac{f_{\mathrm{X}}^{n+1}}{f_{\mathrm{X}}^n}. \tag{40}$$

The quantity E_{X} can then be expanded in powers of Δt.

The predictor coupling scheme, D, is more complicated since, from (37)–(39), the single coupling equation involves quantities at time levels $n+1, n, n-1$ and $n-2$. The single coupling equation also contains quantities such as F_{dd}^{n-1}. These are interpreted as grid-point quantities, evaluated at departure points, which are further interpolated to departure points at a different time level. For the purpose of evaluating the response function, such terms are written as, e.g. $F_{\mathrm{dd}}^{n-1} = F_{\mathrm{d}}^{n-1} \, \mathrm{e}^{-\mathrm{i}kU\Delta t} = F^{n-1} \, \mathrm{e}^{-2\mathrm{i}kU\Delta t}$ in (37)–(39). The response function for scheme D, E_{D}, is then found to satisfy an equation of the form,

$$E_{\mathrm{D}}^3 + AE_{\mathrm{D}}^2 + BE_{\mathrm{D}} + C = 0, \tag{41}$$

where A, B and C are functions of α, β, γ, k, U, the off-centring weights ξ_j and the time-step Δt. Expressions for E_{D} are found, using (41), by expanding A, B and C in powers of Δt, to $\mathrm{O}(\Delta t^2)$, and writing,

$$E_{\mathrm{D}} = E_0 + E_1\Delta t + E_2\Delta t^2 + \mathrm{O}(\Delta t^3). \tag{42}$$

Equating powers of Δt leads to expressions for E_0, E_1 and E_2. There are three roots, one physical and a complex-conjugate pair representing spurious computational modes. The response function for the computational modes is found to be,

$$E_{\mathrm{D}} = \tfrac{1}{2}\gamma\{\overline{\xi}_4 \pm \sqrt{\overline{\xi}_4(\overline{\xi}_4 - 4\overline{\xi}_7)}\}\Delta t + \mathrm{O}(\Delta t^2), \tag{43}$$

which is due entirely to the convective term $-\gamma F$. The first term on the right-hand side of (43) can be eliminated by choosing $\xi_4 = 1$, but a computational mode of $\mathrm{O}(\Delta t^2)$ remains. Though not shown here, this can be eliminated by further choosing $\xi_7 = 1$.

Table 2 lists expressions for the difference between the expanded free response function of the model equation, (6), and that of each of the coupling schemes (only the physical mode is shown for scheme D). In general the difference for all the schemes is minimized when the off-centring weights ξ_1, ξ_2 and ξ_4 each take the value of $1/2$. This corresponds to centring in time the dynamics, fast boundary-layer and convection processes. Note that this conflicts with the values required for a correct steady-state

* Note that this is equivalent to assuming exact interpolation to evaluate upstream quantities.

TABLE 2. EXPRESSIONS FOR THE DIFFERENCE BETWEEN THE EXPANDED FREE RESPONSE FUNCTION OF THE MODEL EQUATION (6) AND THAT OF EACH OF THE IDEALIZED COUPLING SCHEMES DESCRIBED IN SECTION 3

Scheme	$E_{\text{exact}} - E_{\text{X}}$
A	$[(\frac{1}{2}-\xi_2)\beta^2 + (1-\xi_4)\beta\gamma + (\frac{1}{2}-\xi_4)\gamma^2 - (\frac{1}{2}-\xi_1)\alpha^2$ $+\mathrm{i}\alpha\{(1-\xi_2)\beta + (1-\xi_4)\gamma\}$ $+\mathrm{i}kU\{(1-\xi_2)\beta + (1-\xi_4)\gamma\}]\Delta t^2 + \mathrm{O}(\Delta t^3)$
B	$-\mathrm{i}kU\Delta t + [(\frac{1}{2}-\xi_2)\beta^2 + (1-\xi_4)\beta\gamma + (\frac{1}{2}-\xi_4)\gamma^2 - (\frac{1}{2}-\xi_1)\alpha^2$ $+\mathrm{i}\alpha\{\beta + (1-\xi_4)\gamma\} + \mathrm{i}kU(\beta+\gamma) - (kU)^2/2]\Delta t^2 + \mathrm{O}(\Delta t^3)$
C	$[(\frac{1}{2}-\xi_2)\beta^2 + (1-\xi_2)\beta\gamma + (\frac{1}{2}-\xi_4)\gamma^2 - (\frac{1}{2}-\xi_1)\alpha^2$ $+\mathrm{i}\alpha\{(1-\xi_1-\xi_2)\beta + (1-\xi_1-\xi_4)\gamma\}$ $+\mathrm{i}kU\{(1-\xi_2)\beta + (1-\xi_4)\gamma\}]\Delta t^2 + \mathrm{O}(\Delta t^3)$
D	$[(\frac{1}{2}-\xi_2)\beta^2 + (1-\xi_4)\beta\gamma + (\frac{1}{2}-\xi_4)\gamma^2 - (\frac{1}{2}-\xi_1)\alpha^2$ $+\mathrm{i}\alpha\{(1-\xi_1-\xi_2)\beta + (1-\xi_1-\xi_4)\gamma\}]\Delta t^2 + \mathrm{O}(\Delta t^3)$

solution, as shown in Table 1. None of the idealized NWP coupling schemes can reduce the difference to zero, and therefore they cannot represent the free response function of the model equation to second-order accuracy. For $\xi_1 = \xi_2 = \xi_4 = 1/2$, scheme D has the smallest difference,

$$E_{\text{exact}} - E_{\text{D}} = \tfrac{1}{2}\beta\gamma\,\Delta t^2 + \mathrm{O}(\Delta t^3), \tag{44}$$

and is therefore the closest to second-order accuracy. Note that the accuracy of scheme D is independent of the weights ξ_5 and ξ_7, which appear in the predictor (34). Again, the main disadvantage of scheme D is the possibility of introducing spurious computational modes into the solution.

(*a*) *Accuracy improvements*

Improvements in the accuracy of the free response functions for coupling schemes A, B and C are possible with small modifications. The modifications involve the use of departure-point quantities, rather than grid-point quantities, at time level n when computing the tendencies of the physics processes. For example, in scheme C, F^n_{d} could be used rather than F^n on the right-hand side of (24)–(25). This amounts to averaging the physics processes along the semi-Lagrangian trajectory, as is the case for scheme D (see Wedi 1999 for an application of SLAVEPP to the ECMWF model). The effect is to remove those terms containing the advection velocity, U, in the expressions of Table 2. If this is done, when $\xi_1 = \xi_2 = \xi_4 = 1/2$, the modified scheme C represents the free response function with the same accuracy as that of scheme D, (44). However, both the modified scheme C and scheme D are first-order accurate. In addition, using $\xi_4 = 1/2$ in scheme C and $\xi_2 = 1/2$ in scheme D is inconsistent with the values required for the capture of the correct steady-state solution (see Table 1).

From Table 2, scheme B (mostly parallel-split approach) has a term $-\mathrm{i}kU\Delta t$ in the expression for the difference in the response functions. This arises from the parallel-split treatment of the dynamics and the fast boundary-layer process, (16), when the left-hand side of (16) uses a grid-point value F^n rather than a departure point value F^n_{d}.

By construction the coupling schemes of section 3 treat the slow physics term, R, in an explicit and first-order-accurate manner. A second-order-accurate representation of the slow physics would be preferable, and indeed the predictor scheme, D, may have an improved accuracy for the slow physics term by virtue of its use of information from

earlier time levels, but this is by no means clear. Representing the slow physics with higher accuracy, whilst keeping an explicit differencing, is one of the issues considered in the following section.

6. An improved coupling scheme

Physics–dynamics coupling approaches that can capture correct steady-state solutions, without restrictions on the values of any time-off-centring weights, have a number of advantages. A free choice of time-off-centring weights can allow the coupling scheme to be second-order accurate in time. Moreover, numerical stability requirements are easier to satisfy, and decentring of particular processes can be implemented, whilst retaining the ability to capture steady-state solutions correctly. In general, this task is difficult to accomplish using a time-splitting approach. Direct methods of implicitly coupling all processes at once have been investigated recently by, for example, Reisner *et al.* (2003). This strategy can be effective and avoids issues such as splitting errors. A particular difficulty with direct methods, however, is finding efficient preconditioners when many complex physical parametrizations are present. Thus, for current operational NWP models, the time-split coupling approach appears to be the most straightforward one. With this in mind it is useful to search for improved time-splitting approaches to the physics–dynamics coupling problem.

Cullen (2001) and Cullen and Salmond (2003) promote the use of predictor–corrector coupling schemes as a first iteration towards a scheme that couples implicitly all processes. The Cullen scheme (see Cullen 2001) can be written in terms of the SWC methodology as:

1st iteration

$$\frac{F^{(1)} - F^n_{\mathrm{d}_1}}{\Delta t} = -\mathrm{i}\alpha(\xi_1 F^n + \overline{\xi}_1 F^n_{\mathrm{d}_1}) - \gamma(\xi_4 F^n + \overline{\xi}_4 F^n_{\mathrm{d}_1}) + \xi_3 R^n + \overline{\xi}_3 R^n_{\mathrm{d}_1}, \tag{45}$$

$$\frac{F^{(2)} - F^{(1)}}{\Delta t} = -\beta(\xi_2 F^{(2)} + \overline{\xi}_2 F^n_{\mathrm{d}_1}), \tag{46}$$

$$\frac{F^{(3)} - F^{(2)}}{\Delta t} = -\mathrm{i}\alpha\xi_1(F^{(3)} - F^n), \tag{47}$$

where the departure point, d_1, is evaluated using

$$U^{n+1/2} = \tfrac{1}{2}(U^n + U^n_{\mathrm{d}_1}), \tag{48}$$

and

2nd iteration

$$\frac{F^{(4)} - F^n_{\mathrm{d}_2}}{\Delta t} = -\mathrm{i}\alpha(\xi_1 F^{(3)} + \overline{\xi}_1 F^n_{\mathrm{d}_2}) - \gamma(\xi_4 F^{(3)} + \overline{\xi}_4 F^n_{\mathrm{d}_2}) + \xi_3 R^{(3)} + \overline{\xi}_3 R^n_{\mathrm{d}_2}, \tag{49}$$

$$\frac{F^{(5)} - F^{(4)}}{\Delta t} = -\beta(\xi_2 F^{(5)} + \overline{\xi}_2 F^n_{\mathrm{d}_2}), \tag{50}$$

$$\frac{F^{n+1} - F^{(5)}}{\Delta t} = -\mathrm{i}\alpha\xi_1(F^{n+1} - F^{(3)}), \tag{51}$$

where the updated departure point, d_2, is now evaluated using,

$$U^{n+1/2} = \tfrac{1}{2}(U^{(3)} + U^n_{\mathrm{d}_1}). \tag{52}$$

The single coupling equation for this scheme is,

$$\begin{aligned}\frac{F^{n+1}-F^n_{\mathrm{d}_2}}{\Delta t} = &-\mathrm{i}\alpha\{\xi_1(2F^{n+1}-F^n_{\mathrm{d}_2})+\overline{\xi}_1 F^n_{\mathrm{d}_2}\}-\beta\{\xi_2(2F^{n+1}-F^n_{\mathrm{d}_2})+\overline{\xi}_2 F^n_{\mathrm{d}_2}\}\\ &-\gamma(\xi_4 F^n_{\mathrm{d}_1}+\overline{\xi}_4 F^n_{\mathrm{d}_2})+\xi_3 R^{(3)}+\overline{\xi}_3 R^n_{\mathrm{d}_2}\\ &-\chi_{\mathrm{CUL}_1}-\chi_{\mathrm{CUL}_2}+\chi_{\mathrm{CUL}_3},\end{aligned} \tag{53}$$

where

$$\begin{aligned}\chi_{\mathrm{CUL}_1} \equiv\ &\mathrm{i}\alpha\xi_1[\mathrm{i}\alpha(\xi_1 F^{n+1}+\overline{\xi}_1 F^n_{\mathrm{d}_2})+\beta\{\xi_2(2F^{n+1}-F^n_{\mathrm{d}_2})+\overline{\xi}_2 F^n_{\mathrm{d}_2}\}\\ &+\gamma\overline{\xi}_4 F^n_{\mathrm{d}_2}-\xi_3 R^{(3)}-\overline{\xi}_3 R^n_{\mathrm{d}_2}]\Delta t\\ &+\beta\xi_2[\mathrm{i}\alpha\{\xi_1(2F^{n+1}-F^n_{\mathrm{d}_1})+\overline{\xi}_1 F^n_{\mathrm{d}_2}\}+\beta(\xi_2 F^{n+1}+\overline{\xi}_2 F^n_{\mathrm{d}_2})\\ &+\gamma\overline{\xi}_4 F^n_{\mathrm{d}_2}-\xi_3 R^{(3)}-\overline{\xi}_3 R^n_{\mathrm{d}_2}]\Delta t\\ &-\gamma\xi_4\{\mathrm{i}\alpha\overline{\xi}_1 F^n_{\mathrm{d}_1}+\beta\overline{\xi}_2 F^n_{\mathrm{d}_1}+\gamma(\xi_4 F^n+\overline{\xi}_4 F^n_{\mathrm{d}_1})-\xi_3 R^n-\overline{\xi}_3 R^n_{\mathrm{d}_1}\}\Delta t,\end{aligned} \tag{54}$$

$$\begin{aligned}\chi_{\mathrm{CUL}_2} \equiv\ &\mathrm{i}\alpha\beta\xi_1\xi_2[\mathrm{i}\alpha\{2\xi_1 F^{n+1}+\overline{\xi}_1(F^n_{\mathrm{d}_1}+F^n_{\mathrm{d}_2})\}\\ &+\beta\{2\xi_2 F^{n+1}+\overline{\xi}_2(F^n_{\mathrm{d}_1}+F^n_{\mathrm{d}_2})\}+\gamma\{2\xi_4 F^n+\overline{\xi}_4(F^n_{\mathrm{d}_1}+F^n_{\mathrm{d}_2})\}\\ &-\xi_3(R^{(3)}+R^n)-\overline{\xi}_3(R^n_{\mathrm{d}_1}+R^n_{\mathrm{d}_2})]\Delta t^2,\end{aligned} \tag{55}$$

$$\chi_{\mathrm{CUL}_3} \equiv \alpha^2\beta^2\xi_1^2\xi_2^2(F^{n+1}-F^n)\Delta t^3, \tag{56}$$

are splitting errors.

The predicted value $F^{(3)}$ is regarded as a reasonable approximation to F^{n+1} and, likewise, $R^{(3)}$ is considered to be a reasonable approximation to R^{n+1}. Indeed, $F^{(3)}$ is essentially the value used by the current Met Office UM for F^{n+1}. The updated departure-point calculation can have beneficial effects on the stability and accuracy of a coupling scheme (see Cullen 2001), however the SWC methodology, in its current form, does not capture this issue (see Cordero *et al.* 2005 for a discussion of the impact of the departure-point calculation on accuracy and stability). Moreover, multiple sweeps could allow for a more implicit treatment of the nonlinear terms (e.g. the boundary-layer diffusion, see Kalnay and Kanamitsu 1988) than current coupling approaches.

Note that, for the Cullen scheme, the convective term, $-\gamma F$, is not coupled in an implicit manner regardless of the value of ξ_4. Rather, the time averaging uses a predicted value $F^{(3)}$ in the second sweep. The same is true of the slow physics if $R^{(3)}$ can be considered as a reasonable approximation to R^{n+1}. Thus, it is possible for the slow physics to be approximately centred in time if $\xi_3 = 1/2$.

To investigate the behaviour of the Cullen scheme for a non-advecting steady state, let

$$F^{n+1} = F^n = F^n_{\mathrm{d}_1} = F^n_{\mathrm{d}_2} = F'_{\mathrm{ss}}$$

and

$$R^{(3)} = R^n = R^n_{\mathrm{d}_1} = R^n_{\mathrm{d}_2} = R_k\,\mathrm{e}^{\mathrm{i}kx},$$

so that (53) reduces to,

$$0 = \{(\mathrm{i}\alpha+\beta+\gamma)F'_{\mathrm{ss}} - R_k\,\mathrm{e}^{\mathrm{i}kx}\}\{1+(\mathrm{i}\alpha\xi_1+\beta\xi_2-\gamma\xi_4)\Delta t+2\mathrm{i}\alpha\beta\xi_1\xi_2\Delta t^2\}. \tag{57}$$

Differences in the departure points are ignored. When $F'_{\mathrm{ss}} = F_{\mathrm{ss}}$, (57) is satisfied regardless of the values of any of the time-off-centring weights, which is the first requirement of an improved coupling scheme.

Now, the accuracy of the free response function of the Cullen scheme is determined. Setting $R = 0$ at all points on all time levels in the single coupling equation, (53), and, ignoring differences in the departure points, setting $F^n_{d_1} = F^n_{d_2} = F^n \, e^{-ikU\Delta t}$ etc., then the free response function for the Cullen scheme is,

$$E_{CUL} = \frac{A \, e^{-ikU\Delta t} + (\gamma\xi_4 - i\alpha\beta\xi_1\xi_2\Delta t)^2\Delta t^2}{(1 + i\alpha\xi_1\Delta t)^2(1 + \beta\xi_2\Delta t)^2}, \tag{58}$$

where

$$\begin{aligned} A \equiv 1 &- \{i\alpha(\overline{\xi}_1 - \xi_1) + \beta(\overline{\xi}_2 - \xi_2) + \gamma\}\Delta t \\ &- \{(i\alpha\xi_1 + \beta\xi_2 - \gamma\xi_4)(i\alpha\overline{\xi}_1 + \beta\overline{\xi}_2 + \gamma\overline{\xi}_4) - 2i\alpha\beta\xi_1\xi_2\}\Delta t^2 \\ &- 2i\alpha\beta\xi_1\xi_2(i\alpha\overline{\xi}_1 + \beta\overline{\xi}_2 + \gamma\overline{\xi}_4)\Delta t^3. \end{aligned} \tag{59}$$

Expanding E_{CUL} in powers of Δt produces,

$$\begin{aligned} E_{CUL} = 1 &- (i\alpha + ikU + \beta + \gamma)\Delta t \\ &+ [\xi_2\beta^2 + (\xi_2 + \xi_4)\beta\gamma + \xi_4\gamma^2 - \xi_1\alpha^2 \\ &+ i\alpha\{(\xi_1 + \xi_2)\beta + (\xi_1 + \xi_4)\gamma\} \\ &+ ikU(\beta + \gamma + i\alpha) - (kU)^2/2]\Delta t^2 + O(\Delta t^3). \end{aligned} \tag{60}$$

When $\xi_1 = \xi_2 = \xi_4 = 1/2$, then,

$$E_{CUL} - E_{exact} = O(\Delta t^3), \tag{61}$$

and, therefore, the Cullen scheme can represent the free response function of the model equation to second-order accuracy.

Thus, the Cullen scheme does accomplish the task of obtaining a correct steady-state solution, regardless of the values of the time-off-centring weights, while simultaneously allowing for the possibility of second-order accuracy. The convective process, $-\gamma F$, is never coupled implicitly for any value of ξ_4. The second-order accuracy results from an averaging with the predicted value. Additionally, the slow physics process, R, can have enhanced accuracy when compared with the idealized coupling schemes of section 3.

7. Conclusions

Idealized physics–dynamics coupling schemes, motivated by the time-splitting approaches of existing operational NWP models, have been presented and examined in order to identify their numerical properties, in particular their steady-state behaviour and their order of accuracy. Included in the investigation were approaches that can be classified as: (a) purely sequential-split; (b) mostly parallel-split; (c) mixed sequential/parallel splitting; and (d) sequential splitting with predictors. The mixed sequential/parallel-split coupling scheme C was found to satisfy the requirement that a correct steady-state solution be obtained, regardless of the values of any time-off-centring weights, except that the convective process must be coupled explicitly. The predictor scheme D always couples the convection explicitly, and has some restriction on the differencing of the fast boundary-layer process. However, the flexibility of the predictor allows for a stable and accurate differencing of this process. The remaining schemes, A and B, can capture the correct steady-state solution, but specific values of some of the time-off-centring

weights are required, which then compromise the numerical stability of the coupling approach.

All of the idealized schemes A–D fail to represent the free response function of the model equation with second-order accuracy. Scheme D attempts to improve the accuracy of the slow physics and the convective process by the use of predictor values from previous time levels. Although scheme D does have the highest accuracy, unfortunately spurious computational modes can be introduced into the solution. The accuracy of schemes A–C could be improved by using departure-point values rather than grid point values and averaging, as in the SLAVEPP approach of ECMWF, when calculating the physics tendencies. In this case a modified scheme C has an accuracy equivalent to that of scheme D, without the disadvantage of computational modes. This does assume departure-point quantities can be obtained with good accuracy.

An alternative physics–dynamics coupling scheme has been described and analysed. The multiple-sweep predictor–corrector Cullen scheme can capture the correct steady-state solution, regardless of the values of the time-off-centring weights, whilst simultaneously representing the free response function to second-order accuracy. The scheme does not couple the convection process implicitly, instead a predictor–corrector approach is used to improve accuracy. One disadvantage of the Cullen scheme is the extra cost, since the Helmholtz equation is solved twice. On the other hand, some cost savings are possible. Specifically, the initial guess for the second Helmholtz calculation is likely to be sufficiently good that only a few iterations would be required for a satisfactory solution. It is also possible that, due to increased accuracy, the iterative scheme will produce satisfactory solutions using larger time steps leading to greater efficiency (but too large a time step may lead to divergence). Additionally, as mentioned previously, multiple sweeps allow for accuracy and stability improvements due to recalculation of the departure points in a more stable manner, and a more implicit treatment of nonlinear terms.

ACKNOWLEDGEMENTS

Thanks are due to Drs Mike Cullen, Terry Davies and Michail Diamantakis for various helpful discussions.

REFERENCES

Beljaars, A., Bechtold, P., Köhler, M., Morcrette, J.-J., Tompkins, A., Viterbo, P. and Wedi, N. 2004 'The numerics of physical parametrization'. Pp. 113–134 in Proceedings of the ECMWF seminar on recent developments in numerical methods for atmosphere and ocean modelling. European Centre for Medium-Range Weather Forecasts, Shinfield Park, Reading, Berkshire RG2 9AX, UK

Caya, A., Laprise, R. and Zwack, P. 1998 Consequences of using the splitting method for implementing physical forcings in a semi-implicit semi-Lagrangian model. *Mon. Weather Rev.*, **126,** 1707–1713

Cordero, E., Wood, N. and Staniforth, A. 2005 Impact of semi-Lagrangian trajectories on the discrete normal modes of a non-hydrostatic vertical column model. *Q. J. R. Meteorol. Soc.*, **131,** 93–108

Cullen, M. J. P. 2001 Alternative implementations of the semi-Lagrangian semi-implicit schemes in the ECMWF model. *Q. J. R. Meteorol. Soc.*, **127,** 2787–2802

Cullen, M. J. P. and Salmond, D. J. 2003 On the use of a predictor-corrector scheme to couple the dynamics with the physical parametrizations in the ECMWF model. *Q. J. R. Meteorol. Soc.*, **129,** 1217–1236

Dubal, M., Wood, N. and Staniforth, A. 2004 Analysis of parallel vs. sequential splittings for time-stepping physical parameterizations. *Mon. Weather Rev.*, **132,** 121–132

Dubal, M., Wood, N. and Staniforth, A. 2005 Mixed parallel-sequential split schemes for time-stepping multiple physical parameterizations. *Mon. Weather Rev.*, **133,** 989–1002

Kalnay, E. and Kanamitsu, M. 1988 Time schemes for strongly nonlinear damping equations. *Mon. Weather Rev.*, **116,** 1945–1958

Martínez, I. 2004 'Final version of the physics-dynamics coupling'. Pp. 37–47 in Hirlam Newsletter No. 46, available at http://hirlam.knmi.nl, or from HIRLAM-6 Project, c/o Per Undén, SMHI, S-601 76 Norrköping, Sweden

Reisner, J., Wyszogrodzki, A., Mousseau, V. and Knoll, D. 2003 An efficient physics-based preconditioner for the fully implicit solution of small-scale thermally driven atmospheric flows. *J. Comp. Phys.*, **189,** 30–44

Sportisse, B. 2000 An analysis of operator splitting techniques in the stiff case. *J. Comp. Phys.*, **161,** 140–168

Staniforth, A., Wood, N. and Côté, J. 2002a Analysis of the numerics of physics–dynamics coupling. *Q. J. R. Meteorol. Soc.*, **128,** 2779–2799

2002b A simple comparison of four physics–dynamics coupling schemes. *Mon. Weather Rev.*, **130,** 3129–3135

Wedi, N. P. 1999 'The numerical coupling of the physical parametrizations to the 'dynamical' equations in a forecast model'. ECMWF Tech. Memo. 274, European Centre for Medium-Range Weather Forecasts, Shinfield Park, Reading, Berkshire RG2 9AX, UK

Williamson, D. L. 2002 Time-split versus process-split coupling of parameterizations and dynamical core. *Mon. Weather Rev.*, **130,** 2024–2041